ANGULAR DISPLACEMENT IN A SHAFT INVOLVING H-FUNCTION

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Abstract

The object of this paper is to employ Fox's H-function in obtaining a solution of the partial differential equation

$$\frac{\partial^2\theta}{\partial t^2} = b^2\,\frac{\partial^2\theta}{\partial x^2}$$

of angular displacement in a shaft. The result yields a number of particular cases on specializing the parameters and may prove to be useful in several interesting situations.

1. INTRODUCTION:

The H-function of one variable [1, p.10] is defined as:

$$H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right] = (1/2\pi i)\int_L \theta(s)\, x^s\, ds \qquad (1)$$

where $i = \sqrt{(-1)}$,

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}$$

where

$$\sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j \equiv M > 0, \qquad (2)$$

and $|\arg x| < \frac{1}{2}M\pi$.

2. INTEGRAL:

The integral to be evaluated here is

$$\int_0^\mu \cos\left(\frac{\pi\epsilon x}{\mu}\right)\left(\sin\frac{\pi x}{2\mu}\right)^{2\epsilon-\sigma-1}\left(\cos\frac{\pi x}{2\mu}\right)^{\sigma-1} H^{m,n}_{p,q}\left[z\left(\tan\frac{\pi x}{2\mu}\right)^{2h}\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]dx$$

$$= \frac{\mu\, 2^{2\epsilon-\sigma}}{\sqrt{\pi}\,\Gamma(2\epsilon)} H^{m+1,n+1}_{p+2,q+1}\left[z/4^h\left|\begin{matrix}\left(1-\epsilon+\frac{\sigma}{2},h\right),(a_j,\alpha_j)_{1,p},\left(\frac{1}{2}-\epsilon+\frac{\sigma}{2},h\right)\\(\sigma,2h),(b_j,\beta_j)_{1,q}\end{matrix}\right.\right] \qquad (3)$$

provided that $2\epsilon > Re\left(\sigma - 2h\frac{b_j}{\beta_j}\right) > 0$, $|\arg z| < \frac{1}{2}M\pi$, where Mis given in (2).

Proof: To establish (3), express the H-function as given in (1) into the Mellin-Barnes type contour integrals, then interchange the order of summation and integration, which is justified due to the absolute convergence of the integral involved in the process and then evaluate the inner integral with the help of following result

$$\int_0^{\mu} \cos\left(\frac{\pi \epsilon x}{\mu}\right) \left(\sin\frac{\pi x}{2\mu}\right)^{2\epsilon-\sigma-1} \left(\cos\frac{\pi x}{2\mu}\right)^{\sigma-1} dx$$

$$= \frac{\mu\, 2^{2\epsilon-\sigma}\Gamma\left(\frac{2\epsilon-\sigma}{2}\right)\Gamma(\sigma)}{\sqrt{\pi}\,\Gamma\left(\frac{1-2\epsilon+\sigma}{2}\right)\Gamma(2\epsilon)},\ 2\epsilon > Re(\sigma) > 0 \qquad (4)$$

which can be obtained with a little simplification from the result [2, p.375, 3.634 {eqns(2)}].

Finally, interpreting by virtue of (1), we arrive at the desired result.

3. FORMULATION OF THE PROBLEM:

As an example of the application of the H-function in applied mathematics we shall consider the problem of determining the angular displacement or twist $\theta(x,t)$ in a shaft of circular section with its axis along the x-axis. If the ends x = 0 and x = μ of the shaft are free, the displacement $\theta(x,t)$ due to initial twist must satisfy the boundary value problem

$$\frac{\partial^2\theta}{\partial t^2} = b^2 \frac{\partial^2\theta}{\partial x^2} \qquad (5)$$

$$\frac{\partial}{\partial x}\theta(0,t) = 0, \frac{\partial}{\partial x}\theta(\mu,t) = 0, \frac{\partial}{\partial t}\theta(x,0) \qquad (6)$$

$$\theta(x,0) = f(x) \qquad (7)$$

where b is a constant.

The solution of the problem can be written as Churchill [3, p.125(4)]:

$$\theta(x,t) = \frac{1}{2}a_0 + \sum_{l=1}^{\infty} a_l \cos\frac{\pi l x}{\mu} \cos\frac{l\pi b t}{\mu} \qquad (8)$$

where $a_l (l = 0,1,2, \ldots.)$ are the coefficient in the Fourier cosine series for f(x) in the interval (0,μ).

4. SOLUTION OF THE PROBLEM:

Here we shall consider

$$f(x) = \left(\sin\frac{\pi x}{2\mu}\right)^{2\epsilon-\sigma-1}\left(\cos\frac{\pi x}{2\mu}\right)^{\sigma-1} H_{p,q}^{m,n}\left[z\left(\tan\frac{\pi x}{2\mu}\right)^{2h} \Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right] \qquad (9)$$

If t = 0, then by virtue of (9), we have

$$(\sin\frac{\pi x}{2\mu})^{2\epsilon-\sigma-1}(\cos\frac{\pi x}{2\mu})^{\sigma-1}H_{p,q}^{m,n}[z(\tan\frac{\pi x}{2\mu})^{2h}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\propto_j)_{1,p}}]$$

$$=\frac{1}{2}a_0+\sum_{l=1}^{\infty}a_l\cos\frac{\pi lx}{\mu}. \qquad (10)$$

Now multiplying both side of (10) by $\cos(\frac{\pi\epsilon x}{\mu})$ and integrating with respect to x from 0 to μ, we get

$$\int_0^{\mu}\cos(\frac{\pi\epsilon x}{\mu})(\sin\frac{\pi x}{2\mu})^{2\epsilon-\sigma-1}(\cos\frac{\pi x}{2\mu})^{\sigma-1}H_{p,q}^{m,n}[z(\tan\frac{\pi x}{2\mu})^{2h}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_j,\propto_j)_{1,p}}]dx$$

$$=\frac{1}{2}a_0\int_0^{\mu}\cos(\frac{\pi\epsilon x}{\mu})\,dx+\sum_{l=1}^{\infty}a_l\int_0^{\mu}\cos(\frac{\pi\epsilon x}{\mu})\cos\frac{\pi lx}{\mu}dx. \qquad (11)$$

Now using (3) and the orthogonality property of the cosine function, we have

$$a_l=\frac{2^{2l-\sigma+1}}{\sqrt{\pi}\Gamma(2l)}H_{p+2,q+1}^{m+1,n+1}[z/4^h\Big|_{(\sigma,2h),(b_j,\beta_j)_{1,q}}^{(1-l+\frac{\sigma}{2},h),(a_j,\propto_j)_{1,p},(\frac{1}{2}-l+\frac{\sigma}{2},h)}] \qquad (12)$$

With the help of (8) and (12), we obtained the following solution of the problem:

$$\theta(x,t)=\frac{1}{2^{\sigma}\sqrt{\pi}}\sum_{l=1}^{\infty}\frac{2^{2l+1}}{\Gamma(2l)}\cos\frac{\pi lx}{\mu}\cos\frac{l\pi bt}{\mu}$$

$$\times H_{p+2,q+1}^{m+1,n+1}[z/4^h\Big|_{(\sigma,2h),(b_j,\beta_j)_{1,q}}^{(1-\epsilon+\frac{\sigma}{2},h),(a_j,\propto_j)_{1,p},(\frac{1}{2}-\epsilon+\frac{\sigma}{2},h)}] \qquad (13)$$

provided that $Re(\sigma)>0$, |arg z| < ½ Mπ, where Mis given in (2).

4. SPECIAL CASES:

On specializing the parameters, H-function may be reduced to G-function, Lauricella's functions Legendre functions, Bessel functions, hypergeometric functions, Appell's functions, Kampe de Feriet's functions and several other higher transcendental functions. Therefore the result (7) is of general nature and may reduce to be in different forms, which will be useful in the literature on applied Mathematics and other branches.

References

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