ALGEBRAIC AND ANALYTIC PROPERTIES OF $H_R(P)$

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In this paper, first we give the definition of R-norm information measure and then discuss the algebraic and analytic properties of the R–norm information measure $H_R(P)$ and also it will be summarized in the following two theorems.

First we introduce some notations for convenience. We shall often refer to the set of positive real numbers, not equal to 1. We denote this set by R+ with

$$R^{+} = \{R; R > 0, R \neq 1\}$$

We also define Δ_n as the set of all n-ary probability distributions P = (p_1, p_2 , p_3…., p_n) which satisfy the conditions:

$$p_i \geq 0 , \quad \sum_{i=1}^{n} p_i = 1$$

DEFINITION: The R-norm information of the distribution P is defined for R Є R^+

$$H_R(P) = \frac{R}{R-1}\left[1 - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right]$$

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The R-norm information measure (2.1) is a real function $\Delta_n \to R^+$ defined on Δ_n where $n \geq 2$ and R^+ is the set of positive real numbers .This measure is different from Shannon's entropy, Renyi and Havrda and Charvat and Daroczy .

ALGEBRAIC AND ANALYTIC PROPERTIES OF $H_R(P)$:

The algebraic and analytic properties of the R–norm information measure $H_R(P)$ will be summarized in the following two theorems. First we consider the algebraic properties of the R-norm information measure.

Theorem 1: The R-norm information measure $H_R(P)$ has the following

Algebraic properties:

1. $H_R(P) = H_R(p_1, p_2, ..., p_n)$ is a symmetric function of $(p_1, p_2, ..., p_n)$.
2. $H_R(P)$ is no-expansible i.e. $H_R(p_1, p_2, ..., p_{n0}, 0) = H_R(p_1, p_2, ..., p_{n0})$
3. $H_R(P)$ is decisive i.e. $H_R(1, 0) = H_R(0, 1) = 0$.
4. $H_R(P)$ is non-recursive.
5. $H_R(P)$ is pseudo–additive, i.e. if P and Q are independent then

$$H_R(P,Q) = H_R(P) + H_R(Q) - \frac{R-1}{R} H_R(P) H_R(Q)$$ i.e. $H_R(P)$ is non-additive.

Proof:

(1) To prove $H_R(P)=H_R(p_1,p_2,...,p_n)$ is a symmetric function of p_1, p_2……., p_n.

By definition (2.1), we have

$$H_R(P)=\frac{R}{R-1}\left[1-\left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right]$$

$$=\frac{R}{R-1}\left[1-\left[p_1^{\ R}+p_2^{\ R}.......+p_n^{\ R}\right]^{\frac{1}{R}}\right]$$

Since $\left[\left(\sum_{i=1}^{n} p_i^R\right)\right]$ is a symmetric relation in p_i's.

i.e. $p_1^R + P_2^R + \ldots\ldots + p_n^R$ is same if $p_1, p_2, \ldots\ldots, p_n$ are changed in cyclic order.

$$\Rightarrow H_R(P)=\frac{R}{R-1}\left[1-\left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right] \text{ is a symmetric function.}$$

Hence completes the proof.

(2) To prove $H_R(p_1,p_2,...,p_{n0},0)=H_R(p_1,p_2,...,p_{n0})$

To prove this let us consider

$$H_R(p_1,p_2,...,p_{n0},0)=\frac{R}{R-1}\left[1-\left[p_1^{\ R}+p_2^{\ R}.......+p_{n0}^{\ R}+o^R\right]^{\frac{1}{R}}\right]$$

$$= \frac{R}{R-1}\left[1-\left[p_1{}^R + p_2{}^R \ldots\ldots + p_{n0}{}^R\right]^{\frac{1}{R}}\right]$$

$$= \frac{R}{R-1}\left[1-\left(\sum_{i=1}^{n0} p_i^R\right)^{\frac{1}{R}}\right]$$

$$= H_R(p_1, p_2, \ldots, p_{n0})$$

(3) To prove $H_R(P)$ is decisive i.e. $H_R(1, 0) = H_R(0, 1) = 0$.

By definition, we have

$$H_R(P) = \frac{R}{R-1}\left[1-\left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right]$$

Now If consider only two events,

$$H_R(P) = \frac{R}{R-1}\left[1-\left(p_1^R + p_2{}^R\right)^{\frac{1}{R}}\right]$$ And if we take

$p_1=1$, $p_2=0$,then (2.3) becomes

$$H_R(P) = \frac{R}{R-1}\left[1-\left(1^R + 0^R\right)^{\frac{1}{R}}\right] = \frac{R}{R-1}[1-1] = O \quad \Rightarrow H_R(1, 0) = 0$$

Similarly we can proof that $H_R(0, 1) = 0$

(4) To prove $H_R(P)$ is non-recursive, we have to prove that

$$H_R(p_1 + p_2, p_3, \ldots, p_n) + (p_1 + p_2) H_R\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right) \neq H_R(p_1, p_2, \ldots, p_n)$$

For this first we consider

$$H_R\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\right)=\frac{R}{R-1}\left[1-\left\{\frac{p_1^R+p_2^R}{p_1+p_2}\right\}^{\frac{1}{R}}\right]$$

Multiply both sides by (p_1+p_2) in (2.4), we get

$$(p_1+p_2)H_R\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\right)=(p_1+p_2)\ \frac{R}{R-1}\left[1-\left\{\frac{p_1^R+p_2^R}{p_1+p_2}\right\}^{\frac{1}{R}}\right]$$

$$H_R(p_1+p_2,p_3,...,p_n)\ =\frac{R}{R-1}\left[1-\left\{(p_1+p_3)^R+p_3^R+...+p_n^R\right\}^{\frac{1}{R}}\right]$$

By combining 2, we have

$$H_R(p_1+p_2,\ p_3,...,p_n)+(p_1+p_2)H_R\left(\frac{p_1}{p_1+p_2},\frac{p_2}{p_1+p_2}\right)\neq H_R(p_1,p_2,...,p_n)$$

Thus $H_R(p_1,\ p_2,\ p_3,...,p_n)$ is non-recursive.

(5) To prove $H_R(P,Q)=H_R(P)+H_R(Q)-\frac{R-1}{R}H_R(P)H_R(Q)$

i.e. $H_R(P)$ is non-additive

Proof: Let $A_1,A_2,...,A_n$ and $B_1,B_2,...,B_m$ be the two sets of events associated with probability distributions $P\in\Delta_n$ and $Q\in\Delta_m$.We denote the probability of the joint occurrence of events $A_i=(i=1,2,...,n)$ and $B_j=(j=1,2,...,m)$ on $p(A_i\cap B_j)$.

Then the R-norm information is given by

$$H_R(P*Q)=\frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{n}\sum_{j=1}^{m} p_i^R(A_i\cap B_i)\right\}^{\frac{1}{R}}\right]$$

Since the events considered here are stochastically independent therefore, We have $H_R(P*Q)=\frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{n} p_i^R(A_i)\right\}^{\frac{1}{R}}\left\{\sum_{j=1}^{m} p_j^R(B_j)\right\}^{\frac{1}{R}}\right]$

$$=\frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{n} p_i^R\right\}^{\frac{1}{R}}\left\{\sum_{j=1}^{m} p_j^R\right\}^{\frac{1}{R}}\right]$$

$$=\frac{R}{R-1}-\frac{R}{R-1}\left[\left(1-\frac{R-1}{R}H_R(P)\right)\left(1-\frac{R-1}{R}H_R(Q)\right)\right]$$

$$=H_R(P)+H_R(Q)-\frac{R-1}{R}H_R(P)H_R(Q)$$

Theorem 2: Let $H_R(P)=H_R(p_1,\ p_2,\ ...,\ p_n)$ be the R-norm information measure. Then for $P\in\Delta_n$ and $R\in R^+$ we find

(1) $H_R(P)$ Non-negative.

(2) $H_R(P)\geq H_R(1,0,0,\ldots,0)=0$.

(3) $H_R(P)\leq H_R\left(\frac{1}{n},\frac{1}{n},\ldots..,\frac{1}{n}\right)=\frac{R}{R-1}\left[1-n^{\frac{1-R}{R}}\right]$.

(4) $H_R(P)$ is a monotonic function of P.

(5) $H_R(P)$ is continuous at $R \in R^+$.

(6) $H_R(P)$ is stable in p_i, $i = 1, 2, \ldots, n$.

(7) $H_R(P)$ is small for small probabilities.

(8) $H_R(P)$ is a concave function for all p_i.

(9) $\lim_{R\to\infty} H_R(P) = 1 - \max . p_i$.

Proof:

(1) To prove that $H_R(P) > 0$, we consider the following two cases:

Case I: When $R > 1$, then $p_i^R \le p_i \ \forall\, i \quad \Rightarrow \sum_{i=1}^{n} p_i^R \le \sum_{i=1}^{n} p_i = 1$

$$\Rightarrow \left[\sum_{i=1}^{n} p_i^{\frac{R}{\alpha}}\right]^{\frac{1}{R}} < 1 \qquad\qquad \Rightarrow -\left[\sum_{i=1}^{n} p_i^{\frac{R}{}}\right]^{\frac{1}{R}} \ge 0$$

$$\Rightarrow 1 - \left[\sum_{i=1}^{n} p_i^{\frac{R}{}}\right]^{\frac{1}{R}} \ge 0$$

We know that if R >1, then $\frac{R}{R-1} > 0$

Multiplying both sides of (2.9) by $\frac{R}{R-1}$ we get

$$\Rightarrow \frac{R}{R-1}\left[1 - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right] \ge 0$$

But $\frac{R}{R-1}\left[1-\left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right] = H_R(P)$

$$\Rightarrow H_R(P) = \frac{R}{R-1}\left[1-\left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right] \geq 0 \quad \text{for R >1}$$

Case II: When $0 < R < 1$, then $p_i^R \geq p_i \ \forall \ i$

$$\Rightarrow \sum_{i=1}^{n} p_i^R \geq \sum_{i=1}^{n} p_i = 1. \quad \Rightarrow \left[\sum_{i=1}^{n} p_i^R\right]^{\frac{1}{R}} \geq 1. \quad \Rightarrow -\left[\sum_{i=1}^{n} p_i^R\right]^{\frac{1}{R}} \leq -1$$

$\Rightarrow 1-\left[\sum_{i=1}^{n} p_i^R\right]^{\frac{1}{R}} < 0$ We know $\frac{R}{R-1} < 0$ if 0 < R < 1

Multiplying both sides of (2.10) by $\frac{R}{R-1}$, we get

$\Rightarrow \frac{R}{R-1}\left[1-\left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right] \geq 0$ But

$\frac{R}{R-1}\left[1-\left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right] = H_R(P)$ Thus we have

$$H_R(P) = \frac{R}{R-1}\left[1-\left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right] \geq 0$$

Hence we conclude that $H_R(P)$ is non- negative $\forall \ R$ € R^+

(2) To prove $H_R(1,0,0,0,0,0) = 0$

i.e. if one of the probability is equal to 1 and all others are equal to zero, then

$H_R(P)=0$ By definition, we have

$$H_R(P)=\frac{R}{R-1}\left[1-\left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right]$$ And if we take $p_1=1$ and $p_2=p_3=p_4\ldots\ldots=p_n$, then

Then $\left[\sum_{i=1}^{n} p_i^{\frac{R}{}}\right]=p_1^R+P_2^R+\ldots\ldots+p_n^R=1$, $\left[\sum_{i=1}^{n} p_i^{\frac{R}{}}\right]^{\frac{1}{R}}=1$

$1-\left[\sum_{i=1}^{n} p_i^{\frac{R}{}}\right]^{\frac{1}{R}}=0$, $\Rightarrow \frac{R}{R-1}\left[1-\left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right]=0$ When $p_1=1$ and $p_2=p_3=p_{4..}=p_n=0$

$$\Rightarrow H_R(1,0,0,0,0,0)=0$$

(3) To prove $H_R(P)\le H_R\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)=\frac{R}{R-1}\left[1-n^{\frac{1-R}{R}}\right]$

By definition (2.1), we have $H_R(P)=\frac{R}{R-1}\left[1-\left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right]$

And if we take $p_1=p_2=p_3=p_4\ldots\ldots=p_n=\frac{1}{n}$, then becomes

$$H_R\left(\frac{1}{n},\frac{1}{n},\ldots\ldots\frac{1}{n}\right)=\frac{R}{R-1}\left[1-\left(n\left(\frac{1}{n}\right)^R\right)^{\frac{1}{R}}\right]$$

$$\Rightarrow H_R\left(\frac{1}{n},\frac{1}{n},\ldots\ldots\frac{1}{n}\right)=\frac{R}{R-1}\left[1-\left(n^R\left(\frac{1}{n}\right)^{\frac{1}{R}}\right)\right]$$

$= \frac{R}{R-1}\left[1 - n^{\frac{1-R}{R}}\right]$ Now we prove

$$H_R(P) \le H_R\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$$

Then by Lagrange multipliers, we have

$$\left[\sum_{i=1}^{n} p_i^{\,R}\right]^{\frac{1}{R}} \ge n^{(1-R)/R} \qquad \text{for R >1} \qquad \text{And}$$

$$\left[\sum_{i=1}^{n} p_i^{\,R}\right]^{\frac{1}{R}} \le n^{(1-R)/R} \qquad \text{for } 0 < R < 1$$

Equality holds iff $p_i=1/n$ for all i=1,2,……,n .

Substituting the results obtained in into definition and noting that R/R-1>0 for R > 1 and R/R-1< 0 for 0 < R <1 completes the property (3)Here it is noted that the R-norm information measure is maximal if all probabilities are equal and minimal if one probability is equal to unity and all others are equal to zero.

(4) $H_R(P)$ is monotonic iff $H_R(p, 1-p)$ is non-decreasing on $p \in \left[0, \frac{1}{2}\right]$.

By definition (2.1), we have

$$H_R(p, 1-p) = \frac{R}{R-1}\left[1 - \left\{(1-p)^R + p^R\right\}^{\frac{1}{R}}\right]$$

Let us define the function $G(p)$ by

$$G(p) = 1 - \left[(1-p)^R + p^R\right]^{\frac{1}{R}}$$

Differentiate (2.17) w.r.t 'p' we get

$$\frac{d\,G(p)}{d\,p} = 0 - \frac{1}{R}\left[(1-p)^R + p^R\right]^{\frac{1-R}{R}}\left[-R(1-p)^{R-1} + R(p)^{R-1}\right]$$

Now when R > 1 → $\frac{1}{R} - 1 < 0$

$$\Rightarrow -\frac{1}{R}\left[(1-p)^R + p^R\right]^{\frac{1-R}{R}}\left[-R(1-p)^{R-1} + R(p)^{R-1}\right] \geq 0$$

$$\Rightarrow \frac{d\,G(p)}{d\,p} \geq 0 \quad \text{for} \quad R > 1$$

From (2.16), we note that

$$\frac{d}{d\,p} H_R(p, 1-p) = \left(\frac{R}{R-1}\right)\frac{d\,G(p)}{d\,p}$$

$$\frac{d}{d\,p} H_R(p, 1-p) \geq 0 \quad \text{for} \quad R > 1, \; p \in \left[0, \frac{1}{2}\right]$$

Similarly we prove $\frac{d\,G(p)}{d\,p} \leq 0$ for 0 < R < 1 and $p \in \left[0, \frac{1}{2}\right]$

And when 0 < R < 1, then $\frac{R}{R-1} < 0$

$$\Rightarrow \left(\frac{R}{R-1}\right)\frac{d\,G(p)}{d\,p} \geq 0$$ we have

$$\frac{d}{d\,p} H_R(p, 1-p) = \left(\frac{R}{R-1}\right)\frac{d\,G(p)}{d\,p}$$

From (2.20), we have

$$\frac{d}{d\,p} H_R(p, 1-p) \geq 0$$

Thus $H_R(p, 1-p)$ is a non-decreasing function and hence monotonic function.

(5) We know that $\left[\sum_{i=1}^{n} p_i^R\right]^{\frac{1}{R}}$ is continuous for $R \in [0, \infty)$.

Hence, $H_R(p) = \frac{R}{R-1}\left[1-\left(\sum_{i=1}^{n} p_i^{\frac{R}{\alpha}}\right)^{\frac{\alpha}{R}}\right]$ is also continuous at $R \in R^+$

(6) $H_R(p)$ is stable in p_i, $i = 1, 2, ..., n$.

We know that $H_R(p)$ is expansible.

i.e. $H_R(p, 0) = H_R(p)$

Thus by definition (2.1), we have

$$H_R(p;q) = \frac{R}{R-1}\left[1-\left[p^R + q^R\right]^{\frac{1}{R}}\right]$$

$$\Rightarrow Lt_{q\to 0^+} H_R(p;q) = Lt_{q\to 0^=}\left[\frac{R}{R-1}\left[1-\left[p^R + q^R\right]^{\frac{1}{R}}\right]\right] = H_R(p, 0^+) = H_R(p, 0)$$

Together with (2.21), it follows that $H_R(p)$ is stable in p_i.

(7) From (2.16), it follows that

$$Lt_{q\to o^+} H_R(p, q) = Lt_{q\to o^+} \frac{R}{R-1}\left[1-\left\{p^R + q^R\right\}^{\frac{1}{R}}\right] = 0$$

This proves that $H_R(p)$ is small for small probabilities.

(**8**)To prove $H_R(p)$ concave first we define the concave function.

DEFINITION: A function f over a set *S* is said to be concave if for all choices of $x_1, x_2, ..., x_m \in S$ and for all scalars $\lambda_1, \lambda_2, ..., \lambda_m$ such that $\lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1$, the following holders

$$f\left(\sum_{i=1}^{m} \lambda_i x_i\right) \geq \sum_{i=1}^{m} \lambda_i f(x_i) \qquad (2.23)$$

Here we consider random variable *x* taking its values in the set S and *r* probability distributions over *S* on follows:

$$P_k(x) = \{p_k(x_1), ..., p_k(x_m)\} : p_k(x_i) \geq 0, \sum_{i=1}^{m} p_k(x_i) = 1, k = 1, 2, ..., r$$

Let us define another probability distribution over *S*

$$P_0(x) = \{p_0(x_1), ..., p_0(x_m)\} \exists \forall i's$$

$P_0(x_i) = \sum_{k=1}^{r} \lambda_k p_k(x_i)$, where $\lambda_k's$ are non-negative scalars satisfying $\sum_{k=1}^{r} \lambda_k = 1$

then We have $$D = \sum_{k=1}^{r} \lambda_k H_R(P_k) - H_R(P_0), \ R(>0) \neq 1$$

$H_R(P)$ will be concave if *D* is less than zero for $R(>0) \neq 1$. So we consider

$$D = \sum_{k=1}^{r} \lambda_k H_R(P_k) - H_R(P_0)$$

$$\mathrm{D}=\sum_{k=1}^{r}\lambda_k\left[1-\left\{\sum_{i=1}^{m}p_k^R(x_i)\right\}^{\frac{1}{R}}\right]\frac{R}{R-1}-\frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{m}p_0^R(x_i)\right\}^{\frac{1}{R}}\right]$$

$$\mathrm{D}=\frac{R}{R-1}\left[\left[\sum_{i=1}^{m}\left(\sum_{k=1}^{r}\lambda_k p_k(x_i)\right)^R\right]^{\frac{1}{R}}-\left[\sum_{k=1}^{r}\lambda_k\left(\sum_{i=1}^{m}p_k^R(x_i)\right)^{\frac{1}{R}}\right]\right]$$

Now using the inequality $\left[\sum_{k=1}^{r}a_k x_k\right]^t \gtrless \sum_{k=1}^{r}a_k x_k^t$ according as $t \lessgtr 1$, we have

$$\left[\sum_{k=1}^{r}\lambda_k p_k(x_i)\right]^R \gtrless \left[\sum_{k=1}^{r}\lambda_k p_k^R(x_i)\right]$$ According as $R \lessgtr 1$.Therefore

$$\left[\sum_{i=1}^{m}\left(\sum_{k=1}^{r}\lambda_k p_k(x_i)\right)^R\right] \gtrless \left(\sum_{i=1}^{m}\sum_{k=1}^{r}\lambda_k p_k^R(x_i)\right)$$ according as $R \lessgtr 1$,

$$D_1=\left[\sum_{i=1}^{m}\left(\sum_{k=1}^{r}\lambda_k p_k(x_i)\right)^R\right]^{\frac{1}{R}} \gtrless \left[\sum_{k=1}^{r}\lambda_k\left(\sum_{i=1}^{m}p_k^R(x_i)\right)\right]^{\frac{1}{R}}$$ according as $R \lessgtr 1$. (2.24)

Moreover,

$$\left[\sum_{k=1}^{r}\lambda_k\left(\sum_{i=1}^{m}p_k^R(x_i)\right)\right]^{\frac{1}{R}} \gtrless \left[\sum_{k=1}^{r}\lambda_k\left(\sum_{i=1}^{m}p_k^R(x_i)\right)^{\frac{1^1}{R}}\right]=D_2$$ according as $R \lessgtr 1$. (2.25)

Thus $D_2 \lessgtr D_1$ according as $R \lessgtr 1$, which implies that $D<0$ in view of the sign of

$\frac{R}{R-1}$ according as $R \lessgtr 1$. This proved that $H_R(P)$ is concave function P

(9) For simplicity of notation we set $\max_i p_i = p_k$, Assuming n_0 =1,2,……

Case I: when R >1, then $p_i \leq p_k$

$$\Rightarrow p_i^{R} \leq p_k^{R} \forall 1 \leq i \leq n \Rightarrow \sum_{i=1}^{n0} p_i^{R} \leq \sum_{i-1}^{no} p_k^{R} \qquad \Rightarrow \left(\sum_{i=1}^{n0} p_i^{R} \right) \leq \left(no(p_k^{R}) \right)$$

$$\Rightarrow \left(\sum_{i=1}^{n0} p_i^{R} \right)^{\frac{1}{R}} \leq \left(no(p_k^{R}) \right)^{\frac{1}{R}} = \left(no^{\frac{1}{R}} (p_k^{R})^{\frac{1}{R}} \right) = n_0^{\frac{1}{R}} p_k$$

Thus $\left(\sum_{i=1}^{n0} p_i^{R} \right)^{\frac{1}{R}} \leq \ = n_0^{\frac{1}{R}} p_k$, It is also noted that for $R > 1$

$$\left[\sum_{i=1}^{n_0} p_i^{R} \right]^{\frac{1}{R}} \geq p_k \qquad p_k \leq \left\{ \sum_{i=1}^{n_0} p_i^{R} \right\}^{\frac{1}{R}} \leq n_0^{\frac{1}{R}} p_k$$

By taking limit for R → ∞, on each side of (2.29), we obtain

$$\Rightarrow \sum_{i=1}^{n0} p_i^{R} \leq no(p_k^{R})$$

$$\lim_{R\to\infty} \left\{ \sum_{i=1}^{n_0} p_i^{R} \right\}^{R} = p_k = \max_i . p_i$$ Also we know that $$\lim_{R\to\infty} \frac{R}{R-1} = 1$$

And finally $$\lim_{R\to\infty} H_R(P) = \lim_{R\to\infty} \frac{R}{R-1} \left[1 - \left\{ \sum_{i=1}^{n_0} p_i^{R} \right\}^{\frac{1}{R}} \right] = 1 - \max_i . p_i$$

i.e. $\lim_{R\to\infty} H_R(P) = 1 - \max . p_i$ This completes the proof of theorem 2.

Property (9) is of particular interest since it provides us with a direct interpretation of the value of R which can be chosen. It shows that for increasing R, the probability, say p_k, which has the largest value tends to dominate the R-norm information of the distribution P. Therefore the R-norm information for large values of R seems appropriate for those applications in which we are mainly interested in events with large probability It is interest to relate the R-norm information to the information of order α and of type β and Shannon's information measure .As may be expected this depends on the values of R, α and β

Theorem: Let $H_\alpha(p)$ be the information of order α [], and $H_\beta(p)$ the information of type β Havedra and Charvat, 1967; Daroczy, 1970) .Then for

$\beta = R$ It holds that

$H_R(p) = \frac{R}{R-1}\left[1-\left[1-(1-2^{1-R})H_\beta(p)\right]^{\frac{1}{R}}\right]$ for $\alpha = R$ we have

$H_R(p) = \frac{R}{R-1}\left[1-\exp\left[\frac{1-R}{R}H_\alpha(p)\right]\right]$ Since the definition of the information measure of order α and of type β, given by

$$H_\alpha(p) = \frac{1}{1-\alpha}\log\left[\sum_{i=1}^{n} p_i^{\ \alpha}\right], \alpha > 0, \alpha \neq 1 \qquad \text{And}$$

$H_\beta(p) = \frac{1}{1-2^{1-\beta}}\left[1-\sum_{i=1}^{n} p_i^{\ \beta}\right], \beta > 0, \beta \neq 1$ To prove this theorem, let us consider the R.H.S of i.e. $\frac{R}{R-1}\left[1-\exp\left[\frac{1-R}{R}H_\alpha(p)\right]\right]$

$\frac{R}{R-1}\left[1-\exp\left[\frac{1-R}{R}H_\alpha(p)\right]\right] = \frac{R}{R-1}\left[1-\exp\left[\frac{1-R}{R}\left[\frac{1}{1-\alpha}\log\left[\sum_{i=1}^{n} p_i^{\ \alpha}\right]\right]\right]\right]$ Put $\alpha = R$ we have

$$\frac{R}{R-1}\left[1-\exp\left[\frac{1-R}{R}H_{\alpha}(p)\right]\right] = \frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i^{\ R}\right]^{\frac{1}{R}}\right] = H_R(p)$$ Hence proved

i.e. $\frac{R}{R-1}\left[1-\left[1-(1-2^{1-R})H_{\beta}(P)\right]^{\frac{1}{R}}\right]$

$$\frac{R}{R-1}\left[1-\left[1-(1-2^{1-R})H_{\beta}(P)\right]^{\frac{1}{R}}\right]=\frac{R}{R-1}\left[1-\left[1-(1-2^{1-R})\left[\frac{1}{1-2^{1-\beta}}\left[1-\sum_{i=1}^{n} p_i^{\ \beta}\right]\right]\right]^{\frac{1}{R}}\right]$$

Put $\beta = R$ $=\frac{R}{R-1}\left[1-\left[1-(1-2^{1-R})\left[\frac{1}{1-2^{1-R}}\left[1-\sum_{i=1}^{n} p_i^{\ R}\right]\right]\right]^{\frac{1}{R}}\right]$

$=\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i^{\ R}\right]^{\frac{1}{R}}\right] = H_R(p)$ Thus we have

$\frac{R}{R-1}\left[1-\left[1-(1-2^{1-R})H_{\beta}(P)\right]^{\frac{1}{R}}\right] = H_R(p)$ Hence proved the theorem. Now we discuss the most interesting property of the R-norm information measure

i.e. $\lim_{R\to 1} H_R(p) = H_S(p)$

Where $H_S(p)$ denotes the Shannon's information measure.

Since by definition $H_R(p) = \frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i^{\ R}\right]^{\frac{1}{R}}\right]$

$$\Rightarrow \lim_{R\to 1} H_R(p) = \lim_{R\to 1} \frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i^{\ R}\right]^{\frac{1}{R}}\right] = \frac{0}{0}\text{(form)}\,\mathrm{I}$$

Then by L'Hospital Rule, we have

$$\lim_{R\to1} H_R(p) = \lim_{R\to1}\left[1.\left[1-(\sum_{i=1}^{n} p_i^{\,R})^{\frac{1}{R}}\right] - R\left[0-(\frac{d}{dR}(\sum_{i=1}^{n} P_i^{\,R})^{\frac{1}{R}})\right]\right]$$

If we take $T = \left[\sum_{i=1}^{n} p_i^{\,R}\right]^{\frac{1}{R}}$ Taking log both sides log T= $\frac{1}{R}\log\left[\sum_{i=1}^{n} p_i^{\,R}\right]$

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