HALF-SYMMETRIC HSU-CONNECTIONS

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ABSTRACT. Half-symmetric F-connections have been defined and studied by some authors. Hsu-structure manifolds also play an important role in the theory of structures on manifolds. The aim of the present paper is to study some properties of Hsu-connections in a differentiable manifold.

1. *Preliminaries*

Let M^n be n-dimensional differentiable manifold of class C. Suppose there exist a tensor field F(0) of type (1,1) on the manifold M^n satisfying

$$F^2 = {}^{r}I \qquad (1.1)$$

where

$$= F(X), \qquad (1.2)$$

X is arbitrary vector field and any real or complex number. Then we say that the manifold M^n admits a Hsu-structure.

An affine connection D on the manifold M^n will be called a Hsu-connection if it satisfies

$$(D_X F)(Y) = 0 \qquad (1.3)$$

or equivalently

$$D_X =. \qquad (1.4)$$

If $S(X, Y)$ is a torsion tensor of D, we have

$$S(X, Y) = D_X Y - D_Y X - [X, Y]. \qquad (1.5)$$

Thus if a, b are real numbers,

$$S(ax + by) = abS(X, Y). \qquad (1.6)$$

Also $S(X, Y)$ is skew symmetric i.e.,

$$S(X, Y) = -S(Y, X). \qquad (1.7)$$

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Let us call the Hsu-connection D on the manifold M^n half-symmetric if $S(X, Y)$ satisfies

$$S(X, Y) = S(,) + + . \qquad (1.8)$$

2. *Some results*

In this section, we shall prove some theorems on Hsu-connections.

Theorem 2.1. *Lei D be an arbitrary affine connection on the manifold M^n. Then the connection B given by*

$$BxY = ({}^{r} D_x Y +) + (D_x +)$$
$$+ ({}^{r} +) + (D_x +) \qquad (2.1)$$

where , , , are C functions, is *also Hsu-connection on the manifold M^n.*

Proof: In view of (2.1), we have

B_X= (^{r} D_x+) + (D_x+)
+ (^{r}+) + (+).

By virtue of (1.1), the above equation takes the form

B_X= (^{r} D_x+) + ($^r D_x$Y+)
+ (^{r}+) + (+). (2.2)

Barring both sides of (2.1) and using (1.1), we have

= (^{r} + $^r D_x$) + (+ $^r D_x Y$)
+ (^{r} + ^{r}) + (+ ^{r}) (2.3)

In view of equations (2.2) and (2.3), it follows that

=

Hence B is a Hsu-connection on the manifold M^n.

Theorem 2.2. *Let D be an affine connection on the Hsu-structure manifold M^n. If S(X, Y) is a torsion tensor of D, the connection B given by*

$B_X Y$ = DxY + ($^r S(X,Y)$ +)
+ ($^r S(,Y)$ +)
+ ($S(X,$) +)
+ ($S($,) +) (2.4)

is a Hsu-connection on the manifold M^n.

Proof: Replacing Y by in (2.4) and using equations (1.1) and (1.6), we get

B_X = Dx + ($^r S(X,$) + ^{r})
+ ($^r S(,)$ + ^{r})
+ (^{r} $S(X,$ Y) +)
+ ($^r S($, Y) +) (2.5)

Barring both sides of (2.4) and using equations (1.1) and (1.4), we get

= Dx + (^{r}+ ^{r} $S(X,$))
+ (^{r} + $^r S(,)$)
+ (+ $^r S(X,$ Y))
+ () + ($^r S($, Y)) (2.6)

In view of equations (2.5 and (2.6), it follows that

B_X=

Hence B is a Hsu-connection on the manifold M^n.

Theorem 2.3 *Let D be an affine connection on the Hsu-structure manifold M^n. If S(X, Y) is a torsion tensor of D, the connection B given by*

$B_X Y$ = $D_x Y$ + ($^r S(X, Y)$ +)
+ ($^r S(,Y)$ +)
+ ((X,) +)
+ ($^r S(,)$ +) (2.7)

where , , , , are C functions, is a Hsu-connection on the manifold M^n.

Proof: The proof follows easily by virtue of theorem 2.2 and the fact that D is a Hsu-connection on the manifold M^n.

If N(X, Y) is the Nijenhuis tensor formed with F, then we have

N(X, Y) = [,] - - + (2.8)

for arbitrary vector fields X, Y in the manifold M^n.

If N(X, Y) = 0, the structure is called integrable. Therefore we have the following:

Theorem 2.4. *Let D be a Hsu-connection on the manifold M^n with S(X, Y) as its torsion tensor, then the structure shall be integrable if and only if*

$S(,) + rS(X, Y) = + \quad (2.9)$

Proof: The proof follows easily by using definitions of the torsion tensor $S(X, Y)$, Nijenhuis tensor $N(X, Y)$ and the properties of Hsu-connection D on the manifold M^n.

3. *Half-symmetric Hsu-connections*

Theorem 3.1. *Let D be a half-symmetric Hsu-connection on the manifold M^n. Then the connection B given by (2.1) is also half-symmetric provided that:*

(3.1)

Proof: In view of (2.1), we have

$B_X Y = (^r D_x Y +) + (^r D_x +)$
$+ (^r +) + (^r +).$

By virtue of (1.1) and (1.4), the above equation takes the form

$B_x Y = 2^r D_x Y + 2 + 2^r + \quad (3.2)$

If $s(X, Y)$ is the torsion tensor of the connection B,

$s(X, Y) = B_X Y - B_Y X - [X, Y]. \quad (3.3)$

Substituting the value of $B_X Y$ etc. in (3.3) and on simplification, we get

$s(X, Y) = 2^r \{S(X, Y) + [X, Y]\} + 2\{+ \}$
$+ 2^r \{S(, Y)+[, Y]\}+ 2\{+ \}$
$– [X, Y]$

i.e.,

$s(X, Y) = 2^r S(X, Y) + 2 + 2^r S(, Y)$
$+ 2 + (2r – 1) [(X, Y] + 2$
$+ 2^r(, Y) + 2. \quad (3.4)$

Since D is half-symmetric, hence in view of (1.8) and (3.4) we have

$S(,) + + = 2^r S(X, Y) + 2$
$+ 2^r S(, Y) + + 2$
$+ (2^r–1)\{+ + \}$
$+ 2\{ + + \}$
$+ 2^r \{+ + \}$
$+ 2 \{++ \}. \quad (3.5)$

If – –= $[X, Y]$, the equation (3.5) takes the form

$s(,) + + = 2^r S(X, Y) + 2$
$+ 2^r S(, Y) + 2$
$+ (2^r – 1)[X, Y] + 2$
$+ 2^r [, Y] + 2.$

Hence,

$s[, Y] + + = s(X, Y),$

Thus B is half-symmetric connection on M^n.

Theorem 3.2 *If D is a half-symmetric Hsu-connection on the manifold M^n. Then the connection B given by (2.4) is also half-symmetric provided that:*

(3.6)

Proof: In view of (2.4), we have

$BxY = (^r S(X, Y))$
$+ (^r S(, Y) +)$

$+ (S(X,) +)$

$+ (S(,) +).$

Let $s(X, Y)$ be the torsion tensor of the connections B. Putting the values of $B_X Y$ etc. in the equation (3.3) and applying the fact that
$S(X, Y)$ is skew-symmetric, we get

$$s(X, Y) = \{2^r S(X, Y) + \; -)$$
$$+ \; \{^r S(, Y) - {}^r S(, X) + 2\}$$
$$+ \{S(X,) - S(Y,) + 2\}$$
$$+ \{2S(,) + - \; - [X, Y]\}. \quad (3.7)$$

In view of (3.6), the equation (3.7) takes the form

$$s(X, Y) = 2^r S(X, Y) + 2 + 2$$
$$+ 2S(,) - [X, Y] \quad (3.8)$$

Since D is half-symmetric, therefore in view of equations (1.8) and (3.8), we have

$$s(,) + \; + \; = \; 2^r S(X, Y) + 2$$
$$+ 2 + 2S(,)$$
$$- \{[,] + + \}. \quad (3.9)$$

In view of (3.6), the equation (3.9) takes the form

$$s(,) + \; + \; = s(X, Y).$$

Thus B is half-symmetric connection on M^n.

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