Stability analysis of discrete-time prey-predator system with predator dependent on alternative resources

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Abstract: *In the present study, the stability analysis of discrete-time prey-predator model with predator dependent on alternative resources is examined. Stability of the system at the fixed points has been discussed. The specific conditions for existence of flip bifurcation and Hopf bifurcation have been derived.*

Keywords: Prey-predator system, forward Euler method, fixed points, flip bifurcation, Hopf bifurcation.

A. Introduction

The prey-predator model is an area of great interest for many ecologists and mathematicians. Many researchers have studied the dynamical behavior of the prey-predator system in ecology and contributed to the growth of continuous models for large size populations [1-6]. A number of researchers established that the discrete-time models are more appropriate and provide efficient results as compared to the continuous models for small size populations [7-12]. Keeping in view the available literature, in this paper, we investigate the stability of discrete-time prey-predator system when the predator is dependent onalternative resources.

Consider a prey-predator system of the form

$$\begin{cases} \frac{dx}{dt} = ax(1-x) - bxy, \\ \frac{dy}{dt} = cy + mbxy - dy^2, \end{cases} \qquad (1)$$

where$x\,(t)$ and $y\,(t)$ represent the densities of prey and predator populations respectively; a, b denotes the intrinsic growth rate of prey and predator respectively; c denotes the growth rate of predator due to alternate resources.Further, d denotes competition among individuals of predator species due to overcrowding and m denotes the conversion rate for predator in a particular habitat.

Applying forward Euler's scheme to the system of equations in (1), we get the discrete-time system as follows:

$$\begin{cases} x \to x + \delta[ax(1-x) - bxy], \\ y \to y + \delta(cy + mbxy - dy^2), \end{cases} \tag{2}$$

whereδ is the step size. Shrinking the step size in Euler's method will yield numerical solutions which more accurately approximate the true solution. Numerical solution to the initial-value problem obtained from Euler's method with step size δ and total number of N steps satisfying 0$<\ \delta \le \frac{L}{N}$,where L is the length of the interval.

B. Stability of the fixed points

The fixed points of the system (2) are O(0,0), A(1,0) ,$C\left(0,\frac{c}{d}\right)$and D($x^*$, y^*) , where x^*, y^* satisfy

$$\begin{cases} a(1-x^*) - by^* = 0, \\ c + mbx^* - dy^* = 0, \end{cases} \tag{3}$$

The Jacobian matrix of (2) at the fixed point (x, y) is written as

$$J=\begin{bmatrix} 1+\ \delta\,(a-2ax-by) & -\delta bx \\ \delta mby & 1+\ \delta\,(c+mbx-2dy) \end{bmatrix}.$$

The characteristic equation of the Jacobian matrix is given by

$$\lambda^2 + \mathrm{p}(x,y)\,\lambda + \mathrm{q}\,(x,y) = 0, \tag{4}$$

where

$$\mathrm{p}(x,y) = -\,\mathrm{tr}(J) = -2 - \delta(a - 2ax - by + c + mbx - 2dy),$$

$$\mathrm{q}(x,y) = \det J = [\,1 + \delta\,(a-2ax-by)][\,1 + \delta\,(c+mbx-2dy)] + \delta^2 \mathrm{m} b^2 xy.$$

Now we state a lemma as similar as in [13]:

Lemma 2.1. Let F(λ) = λ^2 + Bλ + C . Suppose F(1) $>$ 0; λ_1 and λ_2 are roots of F(λ) = 0. Then, we have

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ iff F(-1) > 0 and C < 1;

(ii) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) iff F(-1) < 0;

(iii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ iff F(-1)>0 and C> 1;

(iv) λ_1= -1 and $|\lambda_2| \neq 1$ iff F(-1) and B ≠ 0,2;

(v) λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2| = 1$ iff B^2-4C <0 and C = 1.

Let λ_1 and λ_2 be the roots of (4), which are eigen values of the fixed point (x, y). The fixed point (x, y) is a sink or locally asymptotically stable if $|\lambda_1| < 1$ and $|\lambda_2| < 1$. The fixed point (x, y) is a source or locally unstable if $|\lambda_1| > 1$ and $|\lambda_2| > 1$. The fixed point (x, y) is non-hyperbolic if either $|\lambda_1|$ =1 or $|\lambda_2|$ =1. The fixed point (x, y) is a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$).

Proposition 2.2. The fixed point O (0,0) is not a sink, a source if $\delta > 0$, not non-hyperbolic and not asaddle.

The Jacobian matrix of (2) at O (0,0) is given by

$$J = \begin{bmatrix} 1+\delta a & 0 \\ 0 & 1+\delta c \end{bmatrix}.$$

The eigen values are 1 + δa , 1 + δc. Now

(i) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ $then \frac{-2}{a} < \delta < 0$ $and \frac{-2}{c} < \delta < 0$.

Therefore O(0,0) is not a sink.

(ii) $|\lambda_1| > 1$ $and |\lambda_2| > 1$ $then \delta > 0$ $or \delta < \frac{-2}{a}$ and $\delta > 0$ $or \delta < \frac{-2}{c}$.

Therefore O(0,0) is a source if $\delta > 0$.

(iii) $|\lambda_1|$ = 1 or $|\lambda_2|$ =1 then δa = 0 or $\delta = \frac{-2}{a}$ or δ c = 0 or $\delta = \frac{-2}{c}$.

Therefore O(0,0) is not non-hyperbolic .

(iv) $|\lambda_1| > 1$ and $|\lambda_2| < 1$ $or|\lambda_1| < 1$ and $|\lambda_2| > 1$. Then $\delta > 0$ $or\delta < \frac{-2}{a}$ and $\frac{-2}{c} < \delta < 0$ or $\frac{-2}{a} < \delta < 0$ and $\delta > 0$ or $\delta < \frac{-2}{c}$. Which is not possible. Therefore

O (0, 0) is not a saddle.

Proposition 2.3. The fixed point A(1,0) is not a sink, a source if $\delta > \frac{2}{a}$, non-hyperbolic if $\delta = \frac{2}{a}$, a saddle if $0 < \delta < \frac{2}{a}$. At (1,0) the Jacobian matrix of (2) is

$$J = \begin{bmatrix} 1 - \delta a & -\delta b \\ 0 & 1 + \delta(c + mb) \end{bmatrix}$$

The eigen values are 1 –δa , 1 + $\delta(c + mb)$. Here

(i) $|\lambda_1| < 1$ $and|\lambda_2| < 1$ then$\frac{2}{a} < \delta < 0$ and $\frac{-2}{c+mb} < \delta < 0$, which is not possible. Therefore A (1,0) is not a sink.

(ii) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ then $\delta < 0$ $or\delta > \frac{2}{a}$ and $\delta > 0$ $or\delta < \frac{-2}{c+mb}$. Thus A (1, 0) is a source if $> \frac{2}{a}$.

(iii) $|\lambda_1|$ = 1 or $|\lambda_2|$ = 1 then $\delta = 0$ or $\delta = \frac{2}{a}$ or $\delta = 0$ or $= \frac{-2}{c+mb}$. Therefore A (1, 0) is non-hyperbolic if $= \frac{2}{a}$.

(iv) $|\lambda_1| > 1$ and $|\lambda_2| < 1$ or $|\lambda_1| < 1$ and $|\lambda_2| > 1$. Then $\delta < 0$ or $\delta > \frac{2}{a}$ and $\frac{-2}{c+mb} < \delta < 0$ or $\frac{2}{a} > \delta > 0$ and $\delta > 0$ or $\delta < \frac{-2}{c+mb}$. Therefore A(1,0) is a saddle if 0< $\delta < \frac{2}{a}$.

Proposition 2.4. The fixed point C$\left(0, \frac{c}{d}\right)$is a sink if 0< $\delta <$ min $\left\{\frac{2}{c}, \frac{2d}{bc-ad}\right\}$and bc-ad > 0, a source if $\delta >$min $\left\{\frac{2}{c}, \frac{2d}{bc-ad}\right\}$and bc-ad > 0 , non-hyperbolic if $\delta = \frac{2}{c}$ or $\frac{2}{bc-ad}$, a saddle for all values of parameters except the values which are there in former cases.

The Jacobian matrix of (2) at C$\left(0, \frac{c}{d}\right)$is

$$J = \begin{bmatrix} 1+\delta(a-\frac{bc}{d}) & 0 \\ \frac{\delta mbc}{d} & 1-\delta c \end{bmatrix}$$

The eigen values are 1+δ $(a-\frac{bc}{d})$ and 1-δc .

(i) $|\lambda_1| < 1$ and$|\lambda_2| < 1$ implies $\frac{2d}{bc-ad} > \delta > 0$ and $\frac{2}{c} > \delta > 0$. Therefore C$\left(0,\frac{C}{d}\right)$is a sink if 0< $\delta < \min\left\{\frac{2}{c}, \frac{2d}{bc-ad}\right\}$ and bc-ad> 0.

(ii) $|\lambda_1| > 1$and $|\lambda_2| > 1$ then $\delta > 0$ or $\delta > \frac{2d}{bc-ad}$ and $\delta < 0$ or $\delta > \frac{2}{c}$. Therefore C$\left(0,\frac{C}{d}\right)$is a source if $\delta > \min\left\{\frac{2}{c}, \frac{2d}{bc-ad}\right\}$and bc-ad> 0.

(iii) $|\lambda_1| = 1$ or $|\lambda_2| = 1$ then $\delta = 0$ or $\delta = \frac{2d}{bc-ad}$ or $\delta = 0$ or $\delta = \frac{2}{c}$. Thus C$\left(0,\frac{C}{d}\right)$is non-hyperbolic if $\delta = \frac{2}{c}$ or $\frac{2}{bc-ad}$.

(iv) $B\left(0,\frac{C}{d}\right)$is a saddle for all values of parameters except for that which lie in (i) to (iii). Now (iii) of proposition 2.4 implies that the parameters lie in the setF_C={(a,b,c,d,m,δ) : $\delta = \frac{2}{c}$,$\delta \neq \frac{2d}{bc-ad}$ and bc-ad> 0, a,b,c,d,m,$\delta > 0$}

(v) If in proposition 2.4 , (iii) holds then one of the eigen values of the fixed point C$\left(0,\frac{C}{d}\right)$is -1 and magnitude of the other eigen value is not unity. The point C$\left(0,\frac{C}{d}\right)$undergoes flip bifurcation when the parameter changes in small neighborhood of F_C.

The Jacobian matrix of (2) at the fixed point D(x^* , y^*) is given by

$$J = \begin{bmatrix} 1+\delta(a-2ax^*-by^*) & -\delta bx^* \\ \delta mby^* & 1+\delta(c+mbx^*-2dy^*) \end{bmatrix}.$$

The characteristic equation for the above Jacobian matrix is

$$\lambda^2 + p(x^*,y^*)\lambda + q(x^*,y^*) = 0, \quad (5)$$

where p$(x^*,y^*) = -2-\delta(a-2ax^*-by^*+c+mbx^*-2dy^*)$

$= -2 - \delta G,$

q (x^*, y^*) = [1 + $\delta(a - 2ax^* - by^*)$] [$1 + \delta(c + mbx^* - 2dy^*)$] + δ^2m$b^2x^*y^*$

= 1 + $\delta G + \delta^2 H,$

and

G = a$-$2a$x^* - by^* + c + mbx^* - 2dy^*$,

H = ($a - 2ax^* - by^*)\ (c + mbx^* - 2dy^*) + mb^2x^*y^*$.

Now F $(\lambda) = \lambda^2 - (2 + G\delta)\lambda + (1 + G\delta + H\delta^2)$. So F(1) = H$\delta^2$ and F(-1) = 4 +2Gδ+Hδ^2.

The following proposition can be obtained by using the lemma 2.1,

Proposition 2.5. There exist different topological types of D(x^*, y^*) for all possible parameters.

(i) D(x^*, y^*) is a sink if either condition (i.1) or (i.2) holds:

(i.1) $G^2 - 4H \geq 0$ and 0< $\delta < \frac{-G-\sqrt{G^2-4H}}{H}$

$(i.2) G^2 - 4$H< 0 and 0< $\delta < \frac{-G}{H}$.

(ii) D(x^*, y^*) is a source if either condition (ii.1) or (ii.2) holds:

(ii.1) $G^2 - 4H \geq 0$ and$\delta > \frac{-G+\sqrt{G^2-4H}}{H}$,

(ii.2) $G^2 - 4H < 0$ and $\delta > \frac{-G}{H}$.

(iii) D(x^*, y^*) is a non-hyperbolic if either condition (iii.1) or (iii.2) holds

(iii.1) $G^2 - 4H \geq 0$ and $\delta = \frac{-G\pm\sqrt{G^2-4H}}{H}$,

(iii.2) $G^2 - 4H < 0$ and $\delta = \frac{-G}{H}$.

(iv) D (x^*, y^*) is a saddle for all values of the parameters, except for that values which lie in (i) to (iii).

If the condition (iii.1) of the proposition 2.5 holds, then one of the eigen values of the fixed point D(x^*,y^*) is -1 and the magnitude of the other is not unity. The condition (iii.1) of the proposition 2.5 may be expressed as follows:

F_{D1} = { (a,b,c,d,m,δ) : $\delta = \frac{-G-\sqrt{G^2-4H}}{H}$, $G^2 - 4H \geq 0$ and a,b,c,d,m,$\delta > 0$ },

F_{D2} = { (a,b,c,d,m,δ) : $\delta = \frac{-G+\sqrt{G^2-4H}}{H}$, $G^2 - 4H \geq 0$ and a,b,c,d,m,$\delta > 0$ }.

If the term (iii.2) of proposition 2.5 holds, then the eigen values of the fixed point D(x^*, y^*) occur as a conjugate pair of complex numbes with modulus unity. The condition (iii.2) of the proposition 2.5 may be expressed as follows:

H_D = {(a, b, c, d, m,δ):δ=$\frac{-G}{H}$,G^2-4H<0 and a,b,c,d,m,$\delta > 0$ }.

Here F_{D1} and $F_{D1}$are the regions of existence of flip bifurcation, and H_D is region for existence of Hopf bifurcation.

Conclusion

In this paper, the stability analysis of the discrete-time predator-prey system with predator dependent on alternative resources is discussed. We examined the stability of the model at all the fixed points. The maps may undergo flip bifurcation and Hopf bifurcation at the fixed points under specific conditions when δvaries in small neighbourhood of F_{D1} or F_{D2} and H_D respectively.

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