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SOLUTION OF SPACE -TIME FRACTIONAL FOKKER PLANCK EQUATION BY HOMOTOPY ANALYSIS METHOD

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ABSTRACT
In the present paper, we adopt the homotopy analysis method (HAM) to obtain solutions of linear and nonlinear space-time fractional Fokker–Planck equations (FPE). Both the fractional derivatives are described in the Caputo sense. Solution of some particular linear and nonlinear space fractional FPE and time fractional FPE have also been obtained as special cases of our main result.
Key words and phrases: Homotopy analysis method, Fokker-Planck equation, Caputo fractional derivative, Riemann-Liouville fractional integral operator.

1. INTRODUCTION
The Fokker–Planck equation (FPE), first applied to investigate Brownian motion [5] and the diffusion mode of chemical reactions [17], is now largely employed, in various generalized forms, in physics, chemistry, engineering and biology [42]. The FPE arises in kinetic theory [7], where it describes the evolution of the one-particle distribution function of a dilute gas with long-range collisions, such as a Coulomb gas. For some applications of this equation one can refer the works of He and Wu [12], Jumarie [14], Kamitani and Matsuba [15], Xu et al. [47] and Zak [49]. The general FPE for the motion of a concentration field $u(x,t)$ of one space variable x at time t has the form [42]

$$\frac{\partial u}{\partial t}=\left[-\frac{\partial}{\partial x}A(x)+\frac{\partial^2}{\partial x^2}B(x)\right]u(x,t), \tag{1}$$

with the initial condition given by

$$u(x,0)=f(x), \qquad x\in\mathbb{R}, \tag{2}$$

where $B(x)>0$ is called the diffusion coefficient and $A(x)$ is the drift coefficient. The drift and diffusion coefficients may also depend on time. Mathematically, this equation is a linear second-order partial differential equation of parabolic type. Roughly speaking, it is a diffusion equation with an additional first-order derivative with respect to x.

There is a more general form of Fokker–Planck equation which is called the nonlinear Fokker–Planck equation. The nonlinear Fokker–Planck equation has important applications in various areas such as plasma physics; surface physics, population dynamics, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology and marketing (see [8] and references therein). In the one variable case, the nonlinear FPE is written in the following form

$$\frac{\partial u}{\partial t}=\left[-\frac{\partial}{\partial x}A(x,t,u)+\frac{\partial^2}{\partial x^2}B(x,t,u)\right]u(x,t), \tag{3}$$

with the initial condition given by

$$u(x,0)=f(x), \qquad x\in \mathbb{R}. \tag{4}$$

Due to the vast range of applications of the FPE, a lot of work has been done in order to find the numerical solution of this equation. In this context, the works of Buet et al. [2], Harrison [9], Palleschi et al. [41], Vanaja [45], Yildirim [48] and Zorzano [50] are worth mentioning. Our concern in this paper is to consider the numerical solution of the FPE with space- and time-fractional derivatives of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\left[-\frac{\partial^\beta}{\partial x^\beta}A(x,t,u)+\frac{\partial^{2\beta}}{\partial x^{2\beta}}B(x,t,u)\right]u(x,t), \quad t>0, x>0, \; 0<\alpha,\beta\le 1, \tag{5}$$

where α and β are parameters describing the order of the fractional time- and space derivatives, respectively. The function $u(x,t)$ is assumed to be a causal function of time and space, i.e., vanishing for $t<0$ and $x<0$. The fractional derivatives are considered in the Caputo sense as defined in Section 2. In the case of $\alpha=1$ and $\beta=1$, the fractional FPE (5) reduces to the classical nonlinear FPE given by (3).

The objective of this paper is to extend the application of the homotopy analysis method (HAM) to obtain analytic solutions of the space- and time-fractional Fokker–Planck equations. HAM is a computational method that yields analytical solutions and has certain advantages over standard numerical methods. It is free from rounding off errors as it does not involve discretization, and does not require large computer obtained memory or power. The method introduces the solution in the form of a convergent fractional series with elegantly computable terms. HAM was first proposed by the Chinese mathematician S. -J. Liao [29], for solving linear and nonlinear differential and integral equations. Different from perturbation techniques, the homotopy analysis method does not depend upon any small or large parameters. Besides, it logically contains other solution techniques, such as Adomian's decomposition method [6, 13, 18, 19, 35], homotopy perturbation method [36–39], Lyapunov's artificial small parameter method [33], and the δ-expansion method [16], as proved by Liao [3, 21–27, 29, 46]. Considerable research work has been done recently in applying this method to a class of linear and non-linear equations [1, 10, 11, 20, 28, 44].

2. FRACTIONAL CALCULUS

We give some basic definitions and properties of the fractional calculus theory which are used in this paper.

Definition 2.1. A real function $f(x), x>0$, is said to be in the space $C_\mu, \mu\in\mathbb{R}$ if there exists a real number $p(>\mu)$, such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C[0,\infty)$, and it is said to be in the space C_μ^n if $f^{(n)}(x)\in C_\mu, n\in\mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha>0$, of a function $f\in C_\mu, \mu\ge -1$, is defined as [24]

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,dt, \alpha>0, x>0, \tag{6}$$

$$J^0 f(x)=f(x).$$

Properties of the operator J^α can be found in the books by Miller and Ross [34], Oldham and Spanier [40] and Samko et al. [43]. Here, we mention only the following.

For $f\in C_\mu, \mu\ge -1, \alpha,\beta\ge 0$ and $\gamma>-1$:

(i) $J^\alpha J^\beta f(x)=J^{\alpha+\beta} f(x)$,

(ii) $J^\alpha J^\beta f(x)=J^\beta J^\alpha f(x)$,

(iii) $J^\alpha x^\gamma=\dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}$. (7)

For fractional derivatives, there are mainly two definitions.

Definition 2.3. Riemann-Liouville fractional derivative D^α of order $\alpha>0$ [24] is defined as

$D^{\alpha} f(x) = D^n J^{n-\alpha} f(x) \quad , \; n-1<\alpha\le n, n\in\mathbb{N}$.

The Riemann–Liouville derivative has disadvantage that it does not give zero when operated on a constant. The other fractional derivative defined by Caputo, owns this property and is given below.

Definition 2.4. The Caputo fractional derivative of $f(x)$ is defined as [4]:

$$ {}_*D_x^{\alpha} f(x) = J^{n-\alpha} D^n f(x) \quad n-1<\alpha\le n, n\in\mathbb{N}, x>0, \; f\in C_{-1}^n \tag{8}$$

or

$$ {}_*D_x^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}} dt, & n-1<\alpha<n \\ \dfrac{d^n}{dx^n} f(x), & \alpha = n. \end{cases} \tag{9}$$

We shall need the following basic property of Caputo fractional derivative.

Lemma 2.1. If $n-1<\alpha\le n, n\in\mathbb{N}$, $f\in C_{\mu}^n, \mu\ge -1$, then [43]

$$ J^{\alpha}\left({}_*D_x^{\alpha} f(x)\right) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{x^k}{k!}. \tag{10}$$

3. BASIC IDEA OF HAM

We consider the following differential equation

$$ \mathrm{N}\left[\mathbf{u}(x,t)\right] = 0, \tag{11}$$

where N is a non linear operator, x and t are independent variables, $u(x,t)$ is an unknown function. For simplicity, we ignore all boundary and initial conditions, which can be treated in a similar way. By means of generalizing the traditional homotopy method, Liao [31] constructed the so-called zero-order deformation equation as

$$ (1-q)\mathrm{L}\left[\Phi(x,t;q) - u_0(x,t)\right] = qhH(x,t)\mathrm{N}\left[\Phi(x,t;q)\right], \tag{12}$$

where $q\in[0,1]$ is the embedding parameter, $h\neq 0$ is called the convergence contol parameter, $H(x,t)$ is non-zero auxiliary function, L is an auxiliary linear operator with the following property

$\mathrm{L}\left[\Phi(x,t)\right] = 0$ when $\Phi(x,t) = 0$,

$u_0(x,t)$ is an initial guess of $u(x,t)$ and $\Phi(x,t;q)$ is an unknown function. It is important, that one has great freedom to choose auxiliary things in HAM.

When $q=0$, the deformation equation (12) becomes

$$ \Phi(x,t;0) = u_0(x,t), \tag{13}$$

and when $q=1$, since $h\neq 0$, the zero-order deformation equation (12) on using (11) gives

$$ \Phi(x,t;1) = u(x,t). \tag{14}$$

Thus according to (13) and (14), as the embedding parameter q increases from 0 to 1, $\Phi(x,t;q)$ varies continuously from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$.

Using the parameter q, we expand $\Phi(x,t;q)$ in Taylor series as follows:

$$ \Phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{15}$$

where

$$ u_m(x,t) = \frac{1}{m!} \left.\frac{\partial^m \Phi(x,t;q)}{\partial q^m}\right|_{q=0}. \tag{16}$$

We assume that the linear operator L, the initial guess $u_0(x,t)$, the convergence control parameter h and the auxiliary function $H(x,t)$ are properly chosen, such that the series (15) is convergent at $q=1$. Now taking $q=1$ in (15) and using (14), we get

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t). \tag{17}$$

Following Liao [30 - 32], differentiating (12) m times with respect to q, then setting $q=0$, and finally dividing by $m!$, we have the so-called mth-order deformation equation

$$\mathrm{L}\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=hH(x,t)\mathcal{R}_m\left(\vec{u}_{m-1}(x,t)\right), \tag{18}$$

Subject to the initial conditions

$$u_m^{(k)}(x,0)=0,\quad k=0,1,2,...,m-1, \tag{19}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathrm{N}\left[\Phi(x,t;q)\right]}{\partial q^{m-1}}\bigg|_{q=0}, \tag{20}$$

$\vec{u}_{m-1}$ stands for the vector $\vec{u}_{m-1}=\{u_0,u_1,u_2,...,u_{m-1}\}$,

and

$$\chi_m=\begin{cases}0, & m=1\\ 1, & m>1\end{cases}. \tag{21}$$

Applying the inverse operator L^{-1} on both side of equation (18), we have

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+h\mathrm{L}^{-1}\left[\mathcal{R}_m\left(\vec{u}_{m-1}(x,t)\right)\right]. \tag{22}$$

In this way, it is easy to obtain $u_m(x,t)$ for $m\geq 1$, thus getting the solution from equation (17).

4. APPLICATIONS

In this section, we shall apply this method for solving linear/nonlinear fractional FPE.

Example 1. Consider the linear space-time fractional FPE

$$_*D_t^{\alpha}u(x,t)=\left[-{}_*D_x^{\beta}+{}_*D_x^{2\beta}\cdot\frac{x^2}{2}\right]u(x,t),\quad t>0,\ x>0, \tag{23}$$

where $_*D_t^{\alpha}$ and $_*D_x^{\beta}$ are Caputo fractional derivatives defined by (8), $0<\alpha,\beta\leq 1$, subject to the initial condition

$$u(x,0)=x. \tag{24}$$

According to the initial condition (24), we take

$$u_0(x,t)=x. \tag{25}$$

We choose the linear operator

$$\mathrm{L}\cong{}_*D_t^{\alpha}, \tag{26}$$

with the property $\mathrm{L}(c)=0$, where c is a constant.

For the given problem, we now define the nonlinear operator as

$$\mathrm{N}\left[\Phi(x,t;q)\right]=\left[{}_*D_t^{\alpha}\Phi(x,t;q)+{}_*D_x^{\beta}\left(x\Phi(x,t;q)\right)-{}_*D_x^{2\beta}\left(\frac{x^2\Phi(x,t;q)}{2}\right)\right]. \tag{27}$$

For the above operator, with assumption $H(x,t)=1$ we construct the zeroth-order deformation equation from (12) as

$$(1-q)\,{}_*D_t^{\alpha}\left[\Phi(x,t;q)-u_0(x,t)\right]=qh\mathrm{N}\left[\Phi(x,t;q)\right], \tag{28}$$

and the mth-order deformation equation is given by

$$_*D_t^{\alpha}\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=h\mathcal{R}_m\left(\vec{u}_{m-1}(x,t)\right), \tag{29}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = {}_*D_t^{\alpha}(u_{m-1}) + {}_*D_x^{\beta}(xu_{m-1}) - {}_*D_x^{2\beta}\left(\frac{x^2 u_{m-1}}{2}\right). \tag{30}$$

Now applying the operator J_t^{α} on both sides of equation (30) and using (10), we get

$$u_m(x,t) = (\chi_m + h)u_{m-1}(x,t) - (\chi_m + h)u_{m-1}(x,0) + hJ_t^{\alpha}\left[{}_*D_x^{\beta}(xu_{m-1}) - {}_*D_x^{2\beta}\left(\frac{x^2 u_{m-1}}{2}\right)\right], \tag{31}$$

This is a recurrence relation for $m \geq 1$, which on using (25), will give the values of $u_1, u_2, u_3, ...$ as follows.

$$u_1(x,t) = h\left[-\frac{3x^{3-2\beta}}{\Gamma(4-2\beta)} + \frac{2x^{2-\beta}}{\Gamma(3-\beta)}\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{32}$$

$$\begin{aligned} u_2(x,t) = {} & h(1+h)\left[-\frac{3x^{3-2\beta}}{\Gamma(4-2\beta)} + \frac{2x^{2-\beta}}{\Gamma(3-\beta)}\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h^2\left[-\frac{3\Gamma(5-2\beta)x^{4-3\beta}}{\Gamma(4-2\beta)\Gamma(5-3\beta)}\right. \\ & \left. + \frac{2\Gamma(4-\beta)x^{3-2\beta}}{\Gamma(3-\beta)\Gamma(4-2\beta)} - \frac{\Gamma(5-\beta)x^{4-3\beta}}{\Gamma(3-\beta)\Gamma(5-3\beta)} + \frac{3\Gamma(6-2\beta)x^{5-4\beta}}{2\Gamma(4-2\beta)\Gamma(6-4\beta)}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{33}$$

$$\begin{aligned} u_3(x,t) = {} & h(1+h)^2\left[-\frac{3x^{3-2\beta}}{\Gamma(4-2\beta)} + \frac{2x^{2-\beta}}{\Gamma(3-\beta)}\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h^2(1+h)\left[-\frac{6\Gamma(5-2\beta)x^{4-3\beta}}{\Gamma(4-2\beta)\Gamma(5-3\beta)}\right. \\ & \left. + \frac{4\Gamma(4-\beta)x^{3-2\beta}}{\Gamma(3-\beta)\Gamma(4-2\beta)} - \frac{2\Gamma(5-\beta)x^{4-3\beta}}{\Gamma(3-\beta)\Gamma(5-3\beta)} + \frac{3\Gamma(6-2\beta)x^{5-4\beta}}{\Gamma(4-2\beta)\Gamma(6-4\beta)}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ & + h^3\left[\frac{2\Gamma(4-\beta)\Gamma(5-2\beta)x^{4-3\beta}}{\Gamma(3-\beta)\Gamma(4-2\beta)\Gamma(5-3\beta)} - \left(\frac{3\Gamma(5-2\beta)\Gamma(6-3\beta)}{\Gamma(4-2\beta)\Gamma(5-3\beta)\Gamma(6-4\beta)} + \frac{\Gamma(5-\beta)\Gamma(6-3\beta)}{\Gamma(3-\beta)\Gamma(5-3\beta)\Gamma(6-4\beta)}\right.\right. \\ & \left. + \frac{\Gamma(4-\beta)\Gamma(6-2\beta)}{\Gamma(3-\beta)\Gamma(4-2\beta)\Gamma(6-4\beta)}\right)x^{5-4\beta} + \left(\frac{3\Gamma(6-2\beta)\Gamma(7-4\beta)}{2\Gamma(4-2\beta)\Gamma(6-4\beta)\Gamma(7-5\beta)} + \frac{\Gamma(5-\beta)\Gamma(7-3\beta)}{2\Gamma(3-\beta)\Gamma(5-3\beta)\Gamma(7-5\beta)}\right. \\ & \left.\left. + \frac{3\Gamma(5-2\beta)\Gamma(7-3\beta)}{2\Gamma(4-2\beta)\Gamma(5-3\beta)\Gamma(7-5\beta)}\right)x^{6-5\beta} - \frac{3\Gamma(6-2\beta)\Gamma(8-4\beta)}{4\Gamma(4-2\beta)\Gamma(6-4\beta)\Gamma(8-6\beta)}x^{7-6\beta}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{aligned} \tag{34}$$

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Hence

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots \tag{35}$$

When $h = -1$, we have

$$u(x,t)=x+\left(\frac{3x^{3-2\beta}}{\Gamma(4-2\beta)}-\frac{2x^{2-\beta}}{\Gamma(3-\beta)}\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\left(+\frac{2\Gamma(4-\beta)}{\Gamma(3-\beta)\Gamma(4-2\beta)}x^{3-2\beta}-\frac{3\Gamma(5-2\beta)}{\Gamma(4-2\beta)\Gamma(5-3\beta)}x^{4-3\beta}\right.$$
$$\left.-\frac{\Gamma(5-\beta)x^{4-3\beta}}{\Gamma(3-\beta)\Gamma(5-3\beta)}+\frac{3\Gamma(6-2\beta)x^{5-4\beta}}{2\Gamma(4-2\beta)\Gamma(6-4\beta)}\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\left[-\frac{2\Gamma(4-\beta)\Gamma(5-2\beta)}{\Gamma(3-\beta)\Gamma(4-2\beta)\Gamma(5-3\beta)}x^{4-3\beta}\right.$$
$$+\left(\frac{3\Gamma(5-2\beta)\Gamma(6-3\beta)}{\Gamma(4-2\beta)\Gamma(5-3\beta)\Gamma(6-4\beta)}+\frac{\Gamma(5-\beta)\Gamma(6-3\beta)}{\Gamma(3-\beta)\Gamma(5-3\beta)\Gamma(6-4\beta)}+\frac{\Gamma(4-\beta)\Gamma(6-2\beta)}{\Gamma(3-\beta)\Gamma(4-2\beta)\Gamma(6-4\beta)}\right)x^{5-4\beta} \tag{36}$$
$$-\left(\frac{3\Gamma(6-2\beta)\Gamma(7-4\beta)}{2\Gamma(4-2\beta)\Gamma(6-4\beta)\Gamma(7-5\beta)}+\frac{\Gamma(5-\beta)\Gamma(7-3\beta)}{2\Gamma(3-\beta)\Gamma(5-3\beta)\Gamma(7-5\beta)}+\frac{3\Gamma(5-2\beta)\Gamma(7-3\beta)}{2\Gamma(4-2\beta)\Gamma(5-3\beta)\Gamma(7-5\beta)}\right)x^{6-5\beta}$$
$$\left.+\frac{3\Gamma(6-2\beta)\Gamma(8-4\beta)}{4\Gamma(4-2\beta)\Gamma(6-4\beta)\Gamma(8-6\beta)}x^{7-6\beta}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots.$$

Remark 1. Setting $\alpha=1$, equation (23) reduces to space fractional FPE:

$$\frac{\partial}{\partial t}u(x,t)=\left[-{}_*D_x^{\beta}+{}_*D_x^{2\beta}\cdot\frac{x^2}{2}\right]u(x,t),\quad t>0,\,x>0\,, \tag{37}$$

with solution

$$u(x,t)=x+\left(\frac{3x^{3-2\beta}}{\Gamma(4-2\beta)}-\frac{2x^{2-\beta}}{\Gamma(3-\beta)}\right)t+\left(+\frac{2\Gamma(4-\beta)}{\Gamma(3-\beta)\Gamma(4-2\beta)}x^{3-2\beta}-\frac{3\Gamma(5-2\beta)}{\Gamma(4-2\beta)\Gamma(5-3\beta)}x^{4-3\beta}\right.$$
$$\left.-\frac{\Gamma(5-\beta)x^{4-3\beta}}{\Gamma(3-\beta)\Gamma(5-3\beta)}+\frac{3\Gamma(6-2\beta)x^{5-4\beta}}{2\Gamma(4-2\beta)\Gamma(6-4\beta)}\right]\frac{t^{2}}{2!}+\left[-\frac{2\Gamma(4-\beta)\Gamma(5-2\beta)}{\Gamma(3-\beta)\Gamma(4-2\beta)\Gamma(5-3\beta)}x^{4-3\beta}\right.$$
$$+\left(\frac{3\Gamma(5-2\beta)\Gamma(6-3\beta)}{\Gamma(4-2\beta)\Gamma(5-3\beta)\Gamma(6-4\beta)}+\frac{\Gamma(5-\beta)\Gamma(6-3\beta)}{\Gamma(3-\beta)\Gamma(5-3\beta)\Gamma(6-4\beta)}+\frac{\Gamma(4-\beta)\Gamma(6-2\beta)}{\Gamma(3-\beta)\Gamma(4-2\beta)\Gamma(6-4\beta)}\right)x^{5-4\beta} \tag{38}$$
$$-\left(\frac{3\Gamma(6-2\beta)\Gamma(7-4\beta)}{2\Gamma(4-2\beta)\Gamma(6-4\beta)\Gamma(7-5\beta)}+\frac{\Gamma(5-\beta)\Gamma(7-3\beta)}{2\Gamma(3-\beta)\Gamma(5-3\beta)\Gamma(7-5\beta)}+\frac{3\Gamma(5-2\beta)\Gamma(7-3\beta)}{2\Gamma(4-2\beta)\Gamma(5-3\beta)\Gamma(7-5\beta)}\right)x^{6-5\beta}$$
$$\left.+\frac{3\Gamma(6-2\beta)\Gamma(8-4\beta)}{4\Gamma(4-2\beta)\Gamma(6-4\beta)\Gamma(8-6\beta)}x^{7-6\beta}\right]\frac{t^{3}}{3!}+\cdots,$$

which is same as obtained by Yildirim [48] using homotopy perturbation method.

Remark 2. Setting $\beta=1$, equation (23) reduces to time fractional FPE:

$${}_*D_t^{\alpha}u(x,t)=\left[-\frac{\partial}{\partial x}+\frac{\partial^2}{\partial x^2}\left(\frac{x^2}{2}\right)\right]u(x,t),\quad t>0,\,x>0\,, \tag{39}$$

with solution

$$u(x,t)=x\left[1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}+...\right]. \tag{40}$$

Remark 3. For $\alpha=\beta=1$, equation (23) reduces to classical FPE with solution

$$u(x,t)=x\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+...\right] \tag{41}$$

The results for the approximate solution (35) of equation (23) obtained using the homotopy analysis method, for different values of h,α and β are shown in Figure 1. For $h=-1$, the approximate solution (38) is shown in Figure 2(a) and 2(b). Figures 3 and 4 show the solutions for the special cases $\alpha=1$ and $\beta=1$. It is to be noted that only the second-order term of the homotopy analysis solution is used in evaluating the approximate solutions for all figures.

(a) **(b)**

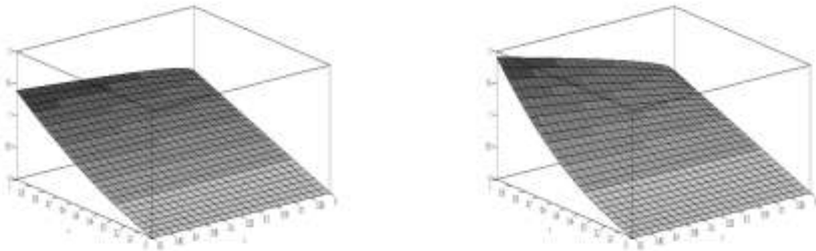

Fig 1. 2[nd] order approximation solution of equation (23) when $\alpha = 0.5, \beta = 0.75$,(a) $h = -0.5$ (b) $h = -1.2$.

(a) **(b)**

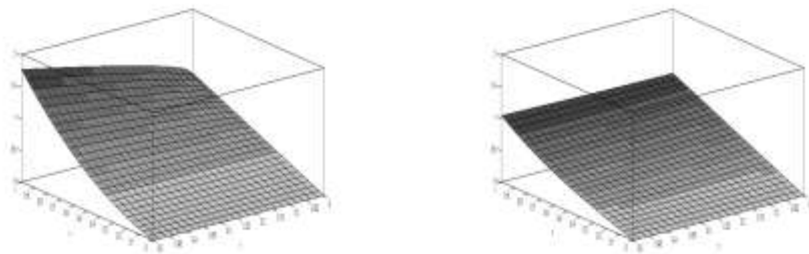

Fig.2. 2[nd] order approximation solution of equation (23) when $h = -1$, (a) $\alpha = 0.5, \beta = 0.75$, (b) $\alpha = 0.75, \beta = 0.5$.

(a) **(b)**

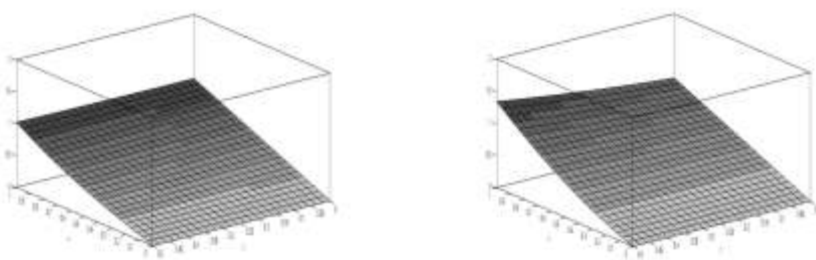

Fig.3. 2[nd] order approximation solution of equation (37) when $\alpha = 1$, (c) $\beta = 0.5$, (d) $\beta = 0.75$.

(a) **(b)**

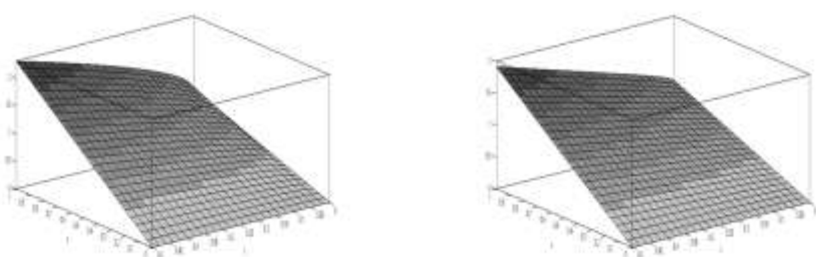

Fig.3. 2[nd] order approximation solution of equation (39) when $\beta = 1$, (a) $\alpha = 0.5$, (b) $\alpha = 0.75$.

Example 2. Consider the nonlinear space-time fractional FPE:

$$ {}_*D_t^{\alpha} u(x,t) = \left[-{}_*D_x^{\beta} \left(\frac{4u}{x} - \frac{x}{3} \right) + {}_*D_x^{2\beta}(u) \right] u(x,t), \quad t>0,\ x>0 \tag{42} $$

where ${}_*D_t^{\alpha}$ and ${}_*D_x^{\beta}$ are Caputo fractional derivatives defined by (8), $0<\alpha,\beta\le 1$, subject to the initial condition

$$ u(x,0) = x^2. \tag{43} $$

According to the initial condition (43), we take

$$ u_0(x,t) = x^2. \tag{44} $$

We choose the linear operator

$$ \mathrm{L} \cong {}_*D_t^{\alpha}, \tag{45} $$

with the property $\mathrm{L}(c)=0$, where c is a constant.

For the given problem, we now define the nonlinear operator as

$$ \mathrm{N}\left[\Phi(x,t;q)\right] = \left[{}_*D_t^{\alpha}\Phi(x,t;q) + {}_*D_x^{\beta}\left(\frac{4\Phi^2(x,t;q)}{x} - \frac{x\Phi(x,t;q)}{3} \right) - {}_*D_x^{2\beta}\left(\Phi^2(x,t;q)\right) \right]. \tag{46} $$

For the above operator, with assumption $H(x,t)=1$ we construct the zeroth-order deformation equation from (12) as

$$ (1-q)\,{}_*D_t^{\alpha}\left[\Phi(x,t;q) - u_0(x,t)\right] = qh\mathrm{N}\left[\Phi(x,t;q)\right], \tag{47} $$

and the mth-order deformation equation is given by

$$ {}_*D_t^{\alpha}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = h\mathcal{R}_m\left(\vec{u}_{m-1}(x,t)\right), \tag{48} $$

where

$$ \mathcal{R}_m(\vec{u}_{m-1}) = {}_*D_t^{\alpha}(u_{m-1}) + {}_*D_x^{\beta}\left(\frac{4}{x}\sum_{i=0}^{m-1} u_i u_{m-1-i} - \frac{x}{3}u_{m-1} \right) - {}_*D_x^{2\beta}\left(\sum_{i=0}^{m-1} u_i u_{m-1-i} \right). \tag{49} $$

Now applying the operator J_t^{α} on both sides of equation (30) and using (10), we get

$$ u_m(x,t) = (\chi_m + h)u_{m-1}(x,t) - (\chi_m + h)u_{m-1}(x,0) + hJ_t^{\alpha}\left[{}_*D_x^{\beta}\left(\frac{4}{x}\sum_{i=0}^{m-1} u_i u_{m-1-i} - \frac{x}{3}u_{m-1} \right) - {}_*D_x^{2\beta}\left(\sum_{i=0}^{m-1} u_i u_{m-1-i} \right) \right], \tag{50} $$

This is a recurrence relation for $m\ge 1$, which on using (44), will give the values of $u_1, u_2, \ldots$ as follows.

$$ u_1(x,t) = h\left[\frac{22}{\Gamma(4-\beta)}x^{3-\beta} - \frac{24}{\Gamma(5-2\beta)}x^{4-2\beta} \right] \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{51} $$

$$ \begin{aligned} u_2(x,t) &= h(1+h)\left[\frac{22}{\Gamma(4-\beta)}x^{3-\beta} - \frac{24}{\Gamma(5-2\beta)}x^{4-2\beta} \right] \frac{t^{\alpha}}{\Gamma(\alpha+1)} + h^2\left[\frac{506}{3}\frac{\Gamma(5-\beta)}{\Gamma(4-\beta)\Gamma(5-2\beta)}x^{4-2\beta} \right. \\ &\left. -184\frac{\Gamma(6-2\beta)}{\Gamma(5-2\beta)\Gamma(6-3\beta)}x^{5-3\beta} - 44\frac{\Gamma(6-\beta)}{\Gamma(4-\beta)\Gamma(6-3\beta)}x^{5-3\beta} + 48\frac{\Gamma(7-2\beta)x^{6-4\beta}}{\Gamma(5-2\beta)\Gamma(7-4\beta)} \right] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{52} $$

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Hence

$$ u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots. \tag{53} $$

In particular, when $h=-1$, we have

$$ \begin{aligned} u(x,t) &= x^2 + \left[-\frac{22}{\Gamma(4-\beta)}x^{3-\beta} + \frac{24}{\Gamma(5-2\beta)}x^{4-2\beta} \right] \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left[\frac{506}{3}\frac{\Gamma(5-\beta)}{\Gamma(4-\beta)\Gamma(5-2\beta)}x^{4-2\beta} \right. \\ &\left. -184\frac{\Gamma(6-2\beta)}{\Gamma(5-2\beta)\Gamma(6-3\beta)}x^{5-3\beta} - 44\frac{\Gamma(6-\beta)}{\Gamma(4-\beta)\Gamma(6-3\beta)}x^{5-3\beta} + 48\frac{\Gamma(7-2\beta)x^{6-4\beta}}{\Gamma(5-2\beta)\Gamma(7-4\beta)} \right] \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots. \end{aligned} \tag{54} $$

Remark 4. Setting $\alpha=1$, equation (42) reduces to space fractional FPE:

$$\frac{\partial}{\partial t}u(x,t)=\left[-{}_*D_x^{\beta}\left(\frac{4u}{x}-\frac{x}{3}\right)+{}_*D_x^{2\beta}(u)\right]u(x,t),\quad t>0, x>0\,, \tag{55}$$

with solution

$$u(x,t)=x^2+\left[-\frac{22}{\Gamma(4-\beta)}x^{3-\beta}+\frac{24}{\Gamma(5-2\beta)}x^{4-2\beta}\right]t+\left[\frac{506}{3}\frac{\Gamma(5-\beta)}{\Gamma(4-\beta)\Gamma(5-2\beta)}x^{4-2\beta}\right.$$
$$\left.-184\frac{\Gamma(6-2\beta)}{\Gamma(5-2\beta)\Gamma(6-3\beta)}x^{5-3\beta}-44\frac{\Gamma(6-\beta)}{\Gamma(4-\beta)\Gamma(6-3\beta)}x^{5-3\beta}+48\frac{\Gamma(7-2\beta)x^{6-4\beta}}{\Gamma(5-2\beta)\Gamma(7-4\beta)}\right]\frac{t^2}{2!}+\cdots. \tag{56}$$

Remark 5. Setting $\beta=1$, equation (42) reduces to time fractional FPE:

$${}_*D_t^{\alpha}u(x,t)=\left[-\frac{\partial}{\partial x}\left(\frac{4u}{x}-\frac{x}{3}\right)+\frac{\partial^2}{\partial x^2}(u)\right]u(x,t),\quad t>0, x>0\,, \tag{57}$$

with solution

$$u(x,t)=x^2\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+...\right),$$

which is same as obtained by Yildirim [48] using homotopy perturbation method.

Remark 6. For $\alpha=\beta=1$, equation (42) reduces to classical nonlinear FPE with solution given as

$$u(x,t)=x^2\left(1+t+\frac{t^2}{2!}+...\right) \tag{59}$$

The results for the approximate solution of equation (42) given by (53) are shown in Figures 5 and 6. Figure 5 shows the variation in solution (53) for different values of h but fixed α and β, whereas Figure 6 represents the solution when $h=-1$ and α and β are varied. It is to be noted that only the second order term of the homotopy solution is used in evaluating the approximate solutions in these figures. Figures 7 and 8 show the solutions for the special cases $\alpha=1$ and $\beta=1$.

(a) (b)

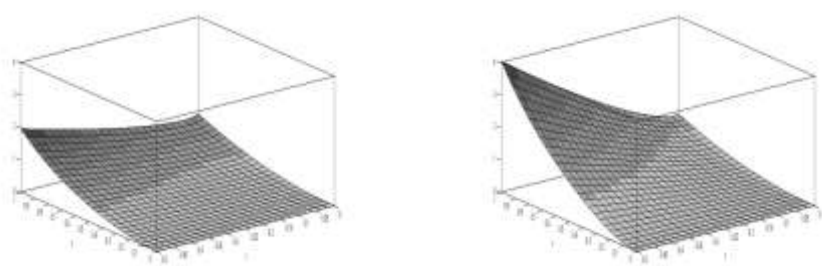

Fig.5. 2nd order approximation solution of equation (42) when $\alpha=0.6, \beta=0.8$,(a) $h=-0.8$ (b) $h=-1.2$

(a) (b)

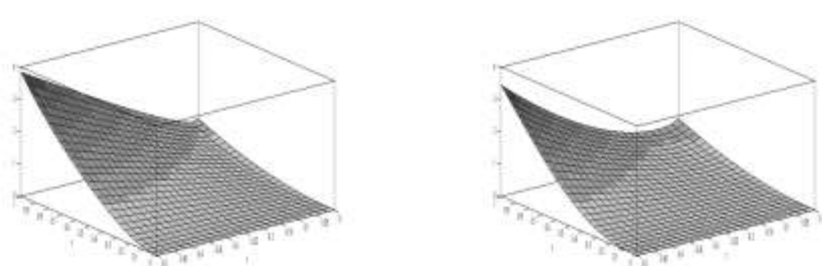

Fig.6. 2nd order approximation solution of equation (42) when $h=-1$, (a) $\alpha=0.5, \beta=0.75$, (b) $\alpha=0.75, \beta=0.5$.

(a) (b)

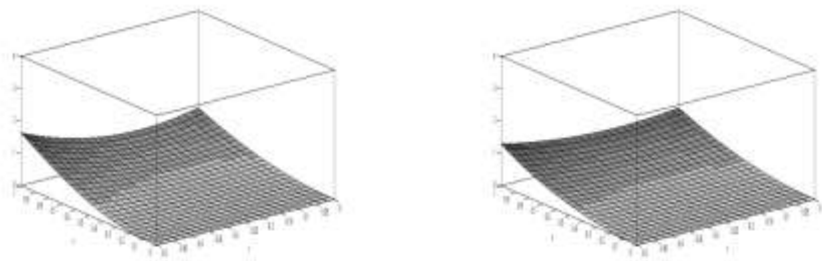

Fig.7. 2^{nd} order approximation solution of equation (55) for (a) $\beta = 0.5$, (b) $\beta = 0.75$.

(a) (b)

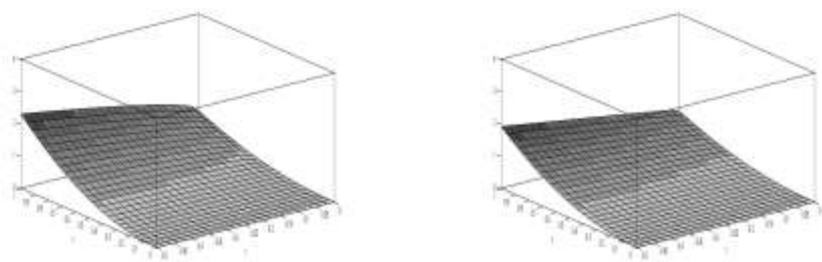

Fig.8. 2^{nd} order approximation solution of equation (58) for (a) $\alpha = 0.5$, (b) $\alpha = 0.75$.

5. CONCLUSIONS

Analytic and approximate solutions of FPE with both space and time Caputo fractional derivatives are obtained using HAM. The solutions are given in the form of a series with easily computable terms. It may be concluded that HAM is a very powerful and efficient technique for solving the model. The study shows that the technique requires less computational work than other existing approaches while supplying quantitatively reliable results.

REFERENCES

[1] Ayub M., Rasheed A., Hayat T., Exact flow of a third grade fluid past a porous plate using homotopy analysis method, *Int. J. Eng. Sci.* **41(18)** (2003), 2091-2103.

[2] Buet, C., Dellacherie, S., Sentis, R., Numerical solution of an ionic Fokker–Planck equation with electronic temperature, *SIAM J. Numer. Anal.* **39 (**2001), 1219–1253.

[3] Cang, J., Tan, Y., Xu, Liao, S. –J., Series solutions of non-linear Riccati differential equations with fractional order, *Chaos Soliton. Fract.*, in press.

[4] Caputo, M., Linear models of dissipation whose Q is almost frequency independent-II, Geophysical Journal International, **13(5)** (1967), 529-539.

[5] Chandrasekhar, S., Stochastic problems in physics and astronomy, *Reviews of Modern Physics* **15 (**1943), 1–89.

[6] Daftardar-Gejji, V., Jafari, H., Adomian decomposition: a tool for solving a system of fractional differential equations, J. *Math. Anal. Appl.* **301 (2)** (2005), 508–518.

[7] Fokker, A., The median energy of rotating electrical dipoles in radiation fields, *Annalen Der Physik* **43** (1914), 810–820.

[8] Frank, T.D., Stochastic feedback, nonlinear families of Markov processes, and nonlinear Fokker–Planck equations, *Physica A* **331** (2004), 391–408.

[9] Harrison, G., Numerical solution of the Fokker–Planck equation using moving finite elements, *Numer. Methods Partial Differ. Eqns* **4 (**1988), 219–232.

[10] Hayat T., Khan M., Ayub M., Couette and Poiseuille flows of an Oldroyd 6-constant fluid with magnetic field, *J. Math. Anal. Appl.* **298(1)** (2004), 225-244.

[11] Hayat T., Khan M., Ayub M., On the explicit analytic solutions of an Oldroyd 6-constant fluid, *Int. J. Eng. Sci.* **42(2)** (2004), 123-135.

[12] He, J. H., Wu, X. H., Construction of solitary solution and compacton-like solution by variational iteration method, *Chaos Solitons Fractals* **29 (**2006), 108–113.

[13] Jafari, H., Gejji, V.D., Solving a system of nonlinear fractional differential equations using Adomain decomposition, *Appl. Math. Comput.* **196** (2006), 644– 651.

[14] Jumarie, G., Fractional Brownian motions via random walk in the complex plane and via fractional derivative, comparison and further results on their Fokker–Planck equations, *Chaos Solitons Fractals* **22 (**2004), 907–925.

[15] Kamitani, Y., Matsuba, I., Self-similar characteristics of neural networks based on Fokker-Planck equation, *Chaos Solitons Fractals* **20** (2004), 329–335.

[16] Karmishin, A.V., Zhukov, A.I., Kolosov, V.G., Method of Dynamics Calculation and Testing for Thin-walled Structures, Mashinostroyenie, Moscow, 1990 (in Russian).

[17] Kramers, H.A., Brownian motion in a field of force and the diffusion model of chemical reactions, *Physica* **7(**1940), 284–304.

[18] Lensic, D., The decomposition method for Cauchy advection-diffusion problems, *Appl. Math. Comput.* **49(4)** (2005), 525–537.

[19] Lensic, D., The decomposition method for initial value problems, *Appl. Math. Comput.* **181** (2006), 206–213.

[20] Liao, S. –J., A new branch of solutions of boundary-layer flows over an impermeable stretched plate, *Int. J. Heat Mass Transfer* **48(12)** (2005), 2529-2539.

[21] Liao, S. –J., An analytic approximate approach for free oscillations of self-excited systems, *Int. J. Nonlinear Mech.* **39 (2)** (2004), 271–280.

[22] Liao, S. -J., An approximate solution technique which does not depend upon small parameters: a special example, *Int. J. Nonlinear Mech.* **32** (1997), 815–822.

[23] Liao, S. –J., An explicit analytic solution to the Thomas–Fermi equation, *Appl. Math. Comput.* **144** (2003), 495–506.

[24] Liao, S. –J., Campo, A., Analytic solutions of the temperature distribution in Blasius viscous flow problems, *J. Fluid Mech*. **453** (2002), 411–425.

[25] Liao, S. –J., Cheung, K.F., Homotopy analysis of nonlinear progressive waves in deep water, *J. Eng. Math.* **45 (2)** (2003), 105–116.

[26] Liao, S. –J., Comparison between the homotopy analysis method and homotopy perturbation method, *Appl. Math. Comput.* **169** (2005), 1186–1194.

[27] Liao, S. –J., On the homotopy analysis method for nonlinear problems, *Appl. Math. Comput.* **147** (2004), 499–513.

[28] Liao, S. –J., Pop I., Explicit analytic solution for similarity boundary layer equations, *Int. J. Heat Mass Transfer* **47** (2004), 75-85.

[29] Liao, S. –J., The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems, Ph.D. Thesis, Shanghai Jiao Tong University, 1992.

[30] Liao, S.-J., Beyond perturbation: a review on the basic ideas of the homotopy analysis method and its applications, *Adv. Mech.* **38** (2008), 1–34.

[31] Liao, S.-J., Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall/CRC Press, Boca Raton, 2003.

[32] Liao, S.-J., Notes on the homotopy analysis method: some definitions and theorems, *Commun. Nonlinear Sci. Numer. Simul.* **14** (2009), 983–997.

[33] Lyapunov, A.M., General Problem on Stability of Motion (English trans.), Taylor and Francis, London, 1992 (Original work published 1892).

[34] Miller, K.S., Ross, B., An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York, 1993.

[35] Momani, S., Al-Khaled, Numerical solutions for systems of fractional differential equations by the decomposition method, *Appl. Math. Comput.* **162 (3)** (2005), 1351–1365.

[36] Momani, S., Odibat, Z., Homotopy perturbation method for nonlinear partial differential equations of fractional order, *Phys. Lett. A* **365 (5-6)** (2007), 345–350.

[37] Momani, S., Odibat, Z., Numerical approach to differential equations of fractional order, *Appl. Math. Comput.* **207** (2007), 96–110.

[38] Momani, S., Odibat, Z., Numerical comparison of methods for solving linear differential equations of fractional order, *Chaos Soliton. Fract.* **31 (5)** (2007), 1248–1255.

[39] Odibat, Z., Momani, S., Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order, *Chaos Soliton. Fract*. **36 (1)** (2008), 167–174.

[40] Oldham, K.B., Spanier, J., The Fractional Calculus. Academic Press, New York, 1974.

[41] Palleschi, V., Sarri, F., Marcozzi, G., Torquati, M. R., Numerical solution of the Fokker–Planck equation: a fast and accurate algorithm, *Phys. Lett. A* **146 (**1990**),** 378–386.

[42] Risken, H., The Fokker–Planck Equation: Method of Solution and Applications, 1989 (Berlin: Springer).

[43] Samko, S.G., Kilbas, A.A., Marichev, O.I., Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, New York, 1993.

[44] Tan Y., Abbasbandy S., Homotopy analysis method for quadratic Riccati differential equation, *Commun. Nonlinear Sci. Numer. Simulat.* **13(3) (**2008), 539–546.

[45] Vanaja, V., Numerical solution of simple Fokker–Planck equation, *Appl. Numer. Math.* **9 (**1992), 533–540.

[46] Wu, Y.Y., Liao, S. –J., Solving the one-loop soliton solution of the Vakhnenko equation by means of the homotopy analysis method, *Chaos Soliton. Fract.* **23(5)** (2004), 1733–1740.

[47] Xu, Y., Ren, F. Y., Liang, J. R., Qiu, W. Y., Stretched Gaussian asymptotic behavior for fractional Fokker–Planck equation on fractal structure in external force fields, *Chaos Solitons Fractals* **20 (**2004), 581–586.

[48] Yildirim, A., Analytical approach to Fokker–Planck equation with space- and time-fractional derivatives by means of the homotopy perturbation method, *Journal of King Saud University (Science)* **22** (2010) , 257–264.

[49] Zak, M., Expectation-based intelligent control, *Chaos Solitons Fractals* **28 (**2006), 616–626.

[50] Zorzano, M. P., Mais, H., Vazquez, L., Numerical solution of two-dimensional Fokker–Planck equations, *Appl. Math. Comput.* **98 (**1999), 109–117.