Applications of Mathematical Models for Blood Flow in a Non-Occluded Artery/Vein of the Cardiovascular System

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ABSTRACT

This studyconsidersblood flow through arteries/veins as similar to flow in a cylindrical vessel. One section of the vessel is assumed to be without a clot and the other section is partly filled with blood clot. The first aim of this study was to consider the applications of mathematical models for the velocity of blood flow along the blood clot in terms of the pressure drop forthree different flow regimes. These were the laminar-laminar, laminar-turbulent and the turbulent-turbulent flow regimes.Graphs of the various mathematical models was plotted using MatLab. It was concluded that for the laminar-laminar flow, the flow velocity is directly proportional to the pressure drop but during the laminar-turbulent flow, flow velocity rises slowly as the pressure drop increases, and in the case of turbulent-turbulent flow, the flow velocity rises sharply from an initial value of 0 to a peak value of 0.19 m/s and continue to increase slowly as the pressure drop increases.

Keywords: non–occluded artery/vein, laminar-laminar flow, laminar-turbulent and the turbulent-turbulent flow, pressure drop

Introduction

The human cardiovascular system is primarily a transport system in which oxygen, carbon dioxide and nutrients are carried by the blood to and from the various muscles and organs. The cardiovascular system consists of two separate parts; the systemic circuit and the pulmonary circuit. These two parts are connected via the heart. From the left ventricle, blood is pumped into the systemic circuit through the aorta (the largest artery in the body). The systemic arteries transport oxygen and nutrients to the various muscles and organs. At the capillary level, oxygen and nutrients diffuse from the vessels into the muscles and organs (Thibodeau and Patton, 1996).When a vein or artery is occluded during surgery, the resistance to the blood flow is increased and as a result a fall in cardiac output is usually observed (Finn*et al*., 2010). Blood clots most often occur in people who cannot move around well or who have had recent surgery or an injury sustained by individuals. When blood clots form within arteries and veins, they obstruct the flow of blood, which are responsible for a number of serious and fatal

conditions such as heart attack or stroke. In cases where turbulence around the blood clot is putting substantial pressure on the blood vessel, internal rupture may occur, which may lead to internal bleeding. A blood clot formed in an artery of the brain can also lead to recurring headaches, speech problems, blurred vision, and aloss of coordination and sense of balance (Ann *et al.*, 2009). Understanding of stenotic flow has proceeded from quite a good number of theoretical, computational and experimental efforts (Smith, 1979; Deshpande *et al*., 1976; Mcdonald, 1979; Ponalgusamy, 2007; Agarwal and Varshney, 2014). Appiah *et al*. (2011) developedmathematical models for the velocity of blood flow in terms of the pressure drop along the clot formed in artery/vein by combining the Poiseuille's equation, the continuity equation, and the Darcy-Weisbach equation for the three different flow regimes. That is, the laminar-laminar, laminar-turbulent and the turbulent-turbulent flow regimes.In their work, Appiah *et al.* (2011) considered the cross-sectional area of the blood vessel against the velocity of the flowing blood. This study reviews the mathematical models developed by Appiah *et al.* and to discuss the relationship between the velocity of the flowing blood and pressure drop. The study also provides some important applications of the models.

Models Formulation

A compartmental model for the human blood circulatory system Figure 1**.**

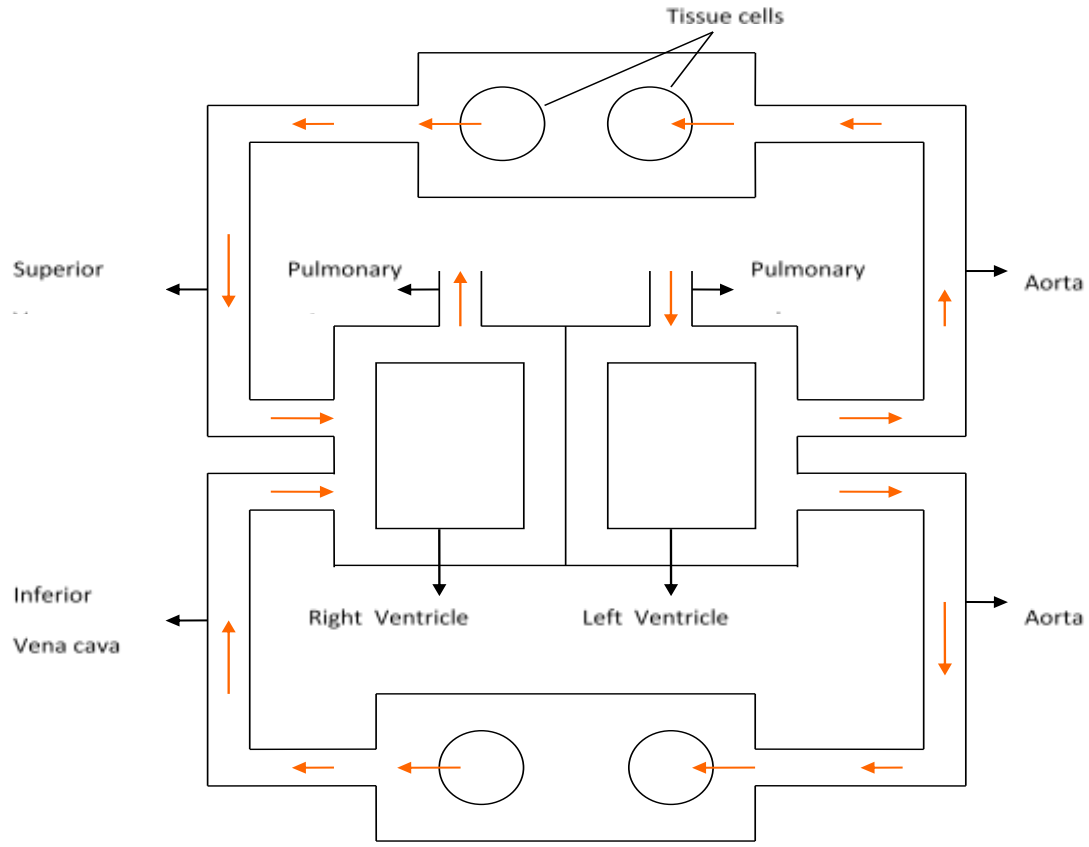

Figure 1 Schematic Diagram of Human Circulatory System

A small part of an artery or vein was taken in a form of a cylinder. This cylindrical vessel has two sections through which blood flows. Blood flows freely through the first part of the vessel but the second section is partially filled with blood clot (Figure 2).

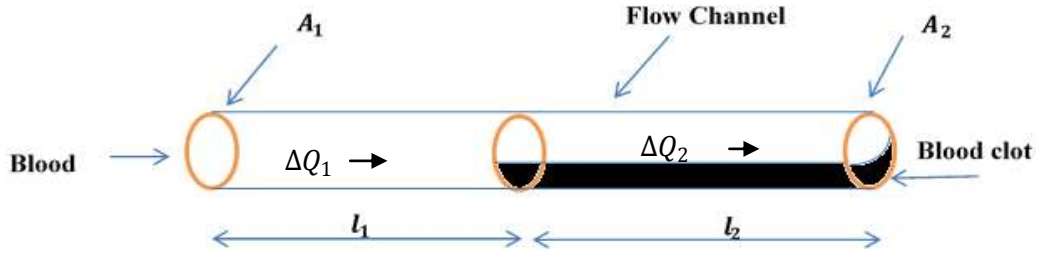

Figure 2Schematic Diagram of Blood Flow Vessel showing two Sections

The first section has a length l_1 and a cross- sectional area,A_1, which represents the normal section of the blood vessel without clot. The second section has a length l_2 and a cross sectional area A_2 which represents the channel along the non-occlusive clot through which the blood is flowing. Also $A_2 < A_1$ and the total pressure drop Δp in both vessel sections is equal to the sum of the pressure drops Δp_1 in the first and Δp_2 in the second section of the clotted vessel.Our models were developed based on some assumptions, especially the assumption that clot behaves like a non-slip solid interface.

In the first regime, the area of the cross-section of the narrowed segment of the vessel is a little smaller but very similar to the cross-section of the normal vessel segment.

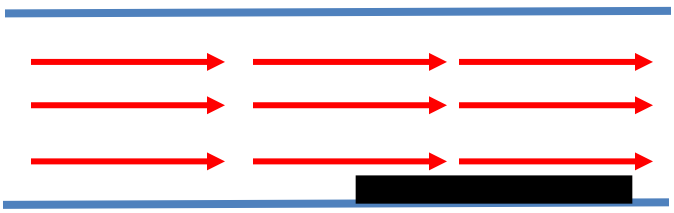

Figure 3 Laminar-Laminar Flow Diagram

The pressure is low and blood flow is slow and laminar in both sections of the vessel. This is called the laminar-laminar flow regime.
According to Appiah *et al.* (2011), the pressure drop Δp_1 in the first vessel segment is given as:

$$\Delta P_1 = 8\pi\eta \frac{v_1 l_1}{A_1} \qquad (1)$$

The pressure drop in the second vessel segment is also given as:

$$\Delta P_2 = 8\pi\eta \frac{v_2 l_2}{A_2} \quad (2)$$

Since the two vessels are connected in series, then: $\Delta p = \Delta p_1 + \Delta p_2$, which gives:

$$\Delta P = 8\pi\eta\left(\frac{v_1 l_1}{A_1} + \frac{v_2 l_2}{A_2}\right) = 8\pi\eta\left(\frac{v_2 A_2 l_1}{A_1^2} + \frac{v_2 l_2}{A_2}\right) \quad (3)$$

Solving Equation (3) for v_2 gives:

$$v_2 = U_L\left(\frac{yM}{y^2 M + 1}\right)\Delta P \quad (4)$$

where $U_L = A_1/(8\pi\eta l_1)$, $M = l_1/l_2$ and $y = A_2/A_1$. Equation (4) is the equation for the laminar-laminar flow.

In the second regime, the area of the cross-section of the narrowed segment of the vessel is about half of the cross-section of the normal vessel segment.

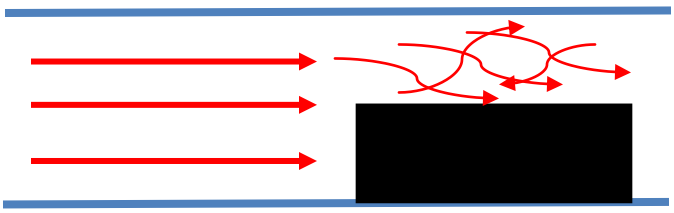

Figure 4 Laminar-Turbulent Flow Diagram

The pressure is somewhat higher so that flow is still relatively slow and laminar in the non-obstructed part of the vessel but is already fast and turbulent in the narrowed section. This is known as the laminar-turbulent regime. Also, the pressure drop Δp_1 in the first vessel segment is the same as Equation (1) and the pressure drop in the second vessel segment is also given as:

$$\Delta P_2 = f\rho \frac{v_2^2 l_2}{\sqrt{A_2}} \quad (5)$$

$\Delta p = \Delta p_1 + \Delta p_2$ and therefore,

$$\Delta P = 8\pi\eta \frac{v_1 l_1}{A_1} + f\rho \frac{v_2^2 l_2}{\sqrt{A_2}} = 8\pi\eta \frac{v_2 A_2 l_1}{A_1^2} + f\rho \frac{v_2^2 l_2}{A_2^{1/2}} \quad (6)$$

Solving Equation (6) for v_2 gives:

$$v_2 = U_{LT}\, M\, y^{3/2}\left(\sqrt{1+\left[\frac{N}{M\, y^{5/2}}\right]\Delta P} - 1\right) \qquad (7)$$

where $U_{LT} = 4\pi\eta / f\rho\sqrt{A_1}$ and $N = \left(f\rho A_1^{3/2}\right)/(16\pi^2\eta^2 l_1)$. Hence Equation (7) is the equation for the laminar-turbulent flow.

The third regime, the clot almost completely blocks that part of the vessel so the area of the cross-section of the narrowed segment of the vessel is far less than the cross-section of the normal vessel segment.

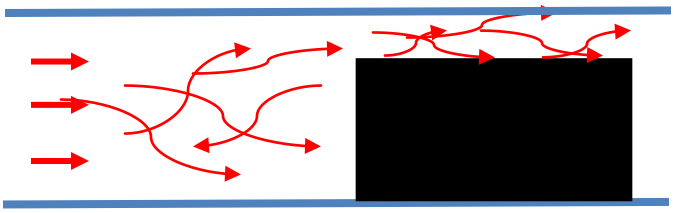

Figure 5 Laminar-Turbulent Flow Diagram

The pressure is very high so that flow is rapid and turbulent in both sections of the vessel. This is called turbulent-turbulent flow regime. The sum of the partial pressure drops in the individual vessels gives the pressure drop in both vessels. This pressure drop is given as:

$$\Delta P = f\rho\frac{v_1^2 l_1}{\sqrt{A_1}} + f\rho\frac{v_2^2 l_2}{\sqrt{A_2}} = f\rho\left(\frac{v_1^2 l_1}{\sqrt{A_1}} + \frac{v_2^2 l_2}{\sqrt{A_2}}\right) \qquad (8)$$

Solving Equation (8) for v_2 gives:

$$v_2 = U_T\left(\frac{1}{\sqrt{y^2 + \frac{1}{M\sqrt{y}}}}\right)\sqrt{\Delta P} \qquad (9)$$

where U_T represent $\sqrt{\sqrt{A_1}/(f\rho l_1)}$. Hence Equation (9) represent the equation for the turbulent-turbulent flow in terms of flow velocity around the clot and pressure drop (Appiah *et al.,* 2011).

ANALYSIS OF RESULTS

Typical parameter values are used in place of the various variables and the analysis of the various models are considered. The area of the normal segment of the vessel is given by:

$$A_1 = \pi R^2 = 3.142(0.015)^2 = 0.0007 \text{ m}^2$$

Table 1 contains typical values of the parameter of blood in an artery and also the values of the cross-sectional area of the flow channel along the clot in terms of laminar-laminar flow, laminar-turbulent flow, and turbulent-turbulent flow.

Table 1 Typical Values of parameters of Blood in an Artery

Parameter	Value	Parameter	Value
Density (ρ)	1060 kg/m^3	Friction coefficient (f)	0.0052
Velocity (v) for laminar flow	0.2 m/s	Area of normal segment (A_1)	0.0007m^2
Velocity (v) for turbulent flow	0.387 m/s	A_2 for laminar-laminar flow $(4/5)A_1$	0.00056m^2
Viscosity (η)	3×10^{-3}	A_2 for laminar-turbulent flow$(1/2)A_1$	0.00035m^2
Length of normal segment (l_1)	3.4 m	A_2 for turbulent-turbulent flow $(1/5)A_1$	0.00014m^2
Length of partly blocked segment (l_2)	2 m	Diameter or artery	0.03 m

(Sources: Cutnelland Johnson, 1998; Glenn, 1998)

Laminar-Laminar Flow Analysis

Now considering that the Equation (4)which is the laminar-laminar flow equation, setting the parameters to typical values of $A_1 = 0.0007m^2$, $A_2 = 0.00056m^2$, $l_1 = 3.4$ m, $l_2 = 2\,m$, and $\eta = 3 \times 10^{-3}\,P$ as in Table 1, the laminar-laminar flow equation reduces to the form:

$$v_2 = 2.12 \times 10^{-4} \Delta p \qquad (10)$$

whereΔp is the pressure drop in mmHg and v_2 is the velocity of blood flow around the clot. To explore the significance of a linear relation, the graph in Figure 6 was drawn using MATLAB.

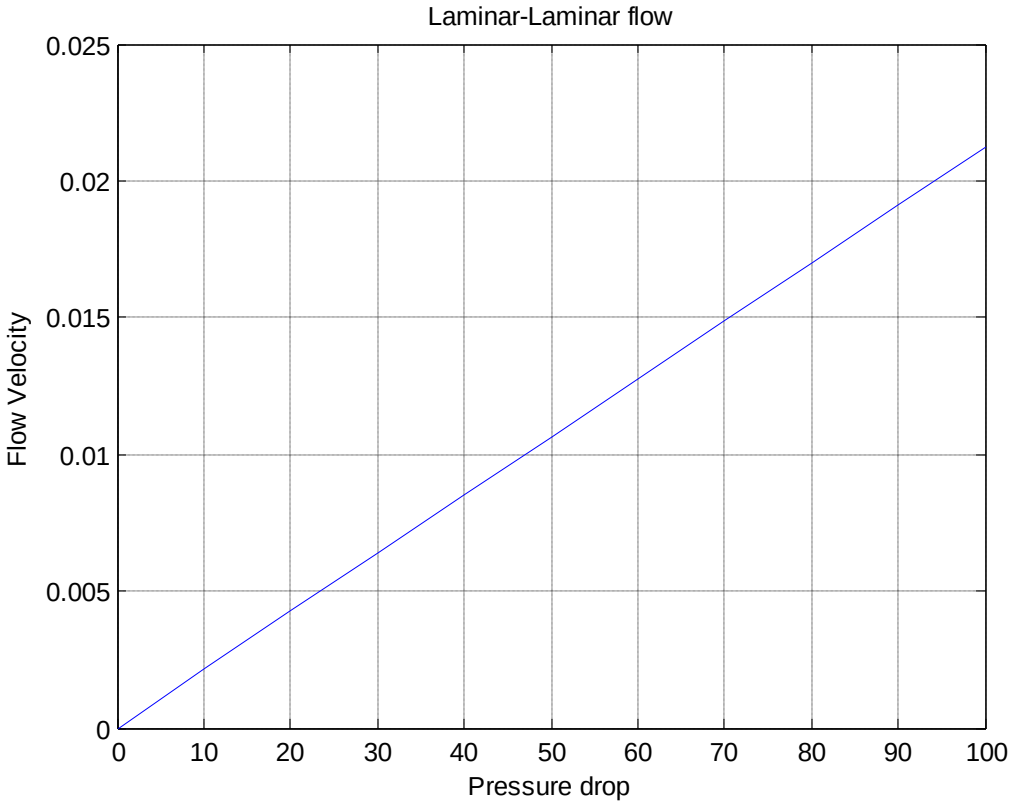

Figure 6Flow Velocity against Pressure for Laminar-Laminar Flow

Figure 4 shows clearly that the function is a linear function and the graph of the function is a straight line as expected. The graph is increasing which means that the flow velocity increases when the pressure increases, and flow velocity decreases when pressure reduces. This is a perfect reflection of the gradient of the line which is positive. Moreover, from the graph, it can be seen that whenever the pressure reduces to zero or disappears completely then the flow velocity also reduces to zero or disappears. When this occurs, it means the heart has stopped working.

Laminar-Turbulent Flow Analysis

Now considering the Equation (7)which is the laminar-turbulent flow equation, setting the parameters to a typical values of $A_1 = 0.0007m^2$, $A_2 = 0.00035m^2$, $l_1 = 3.4\ \mathrm{m}$, $l_2 = 2\ m$, $\rho = 1060\ kg/m^3$, $\mathrm{f} = 0.0052$ and $\eta = 3 \times 10^{-3}\ P$, the laminar-turbulent flow equation reduces to the form:

$$v_2 = -7.5\times10^{-2} + \sqrt{5.63\times10^{-3} + 1.88\times10^{-2}\Delta p} \qquad (11)$$

whereΔp is the pressure drop in millimeter of mercury (mmHg) and v_2 is the velocity of blood flow around the clot. To explore the significance of a linear relation, the graph in Figure 7 was drawn using MATLAB.

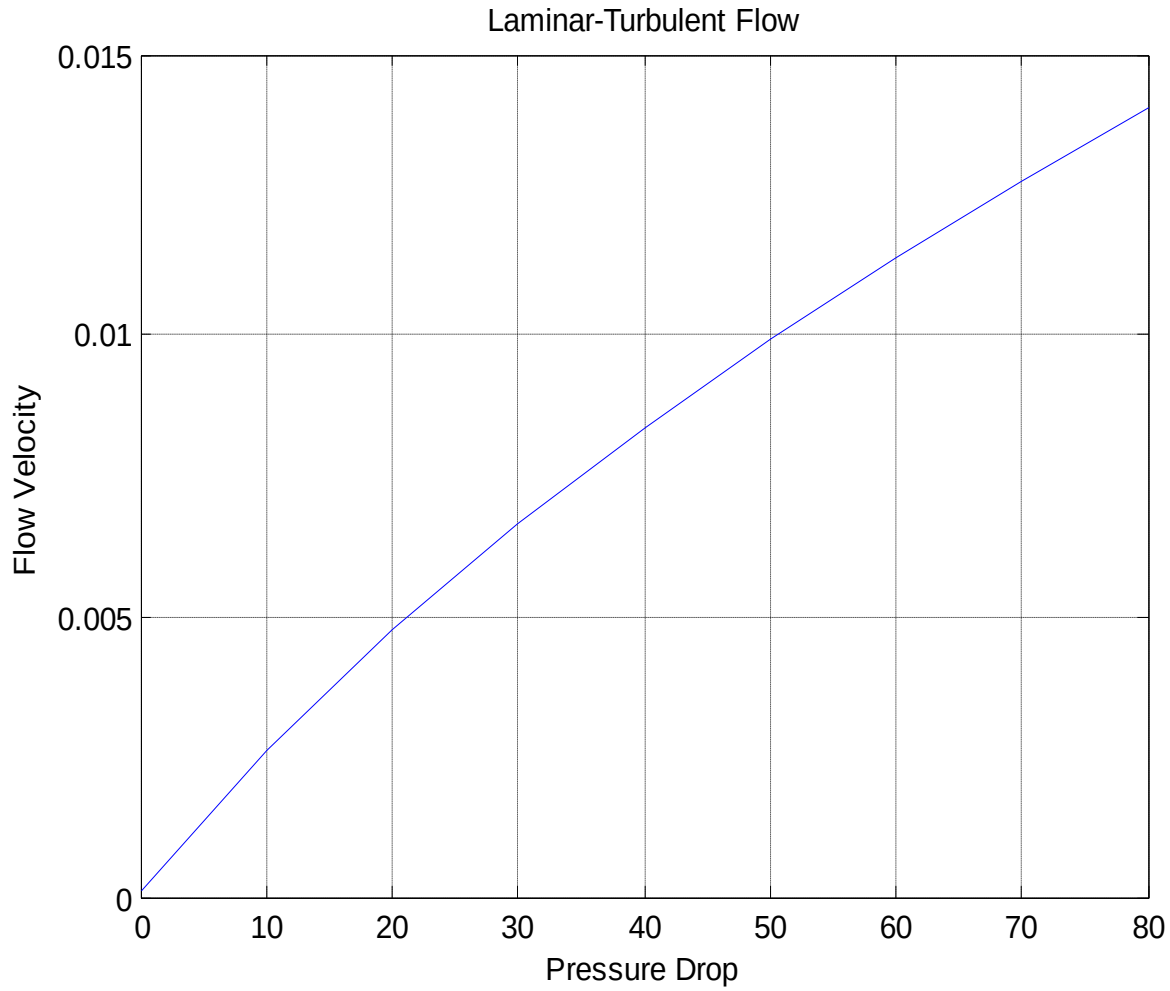

Figure 7Flow Velocity against Pressure for Laminar-Turbulent Flow

The graph is nonlinear because in theory the pressure drop is proportional to the square of the flow velocity. Thus the curve conforms to the theoretical variation. In a blood vessel without the clot, the graph passes through the origin. However vessels with the clot show an intercept on the velocity axis, which implies that the deviation is due to the blockage.

Turbulent -Turbulent Flow Analysis

Now considering the Equation (9)which is the laminar-turbulent flow equation, setting theparameters to a typical values of $A_1 = 0.0007m^2$, $A_2 = 0.00014m^2$, $l_1 = 3.4$ m, $l_2 = 2\ m$, ρ $= 1060\ kg/m^3$, and $\mathrm{f} = 0.0052$, the turbulent-turbulent flow equation reduces to the form

$$v_2 = 3.52 \times 10^{-2} \sqrt{\Delta p} \qquad (12)$$

whereΔp is the pressure drop in millimeter of mercury (mmHg) and v_2 is the velocity of blood flow around the clot. To explore the significance of a linear relation, the graph in Figure 8 was drawn using MATLAB.

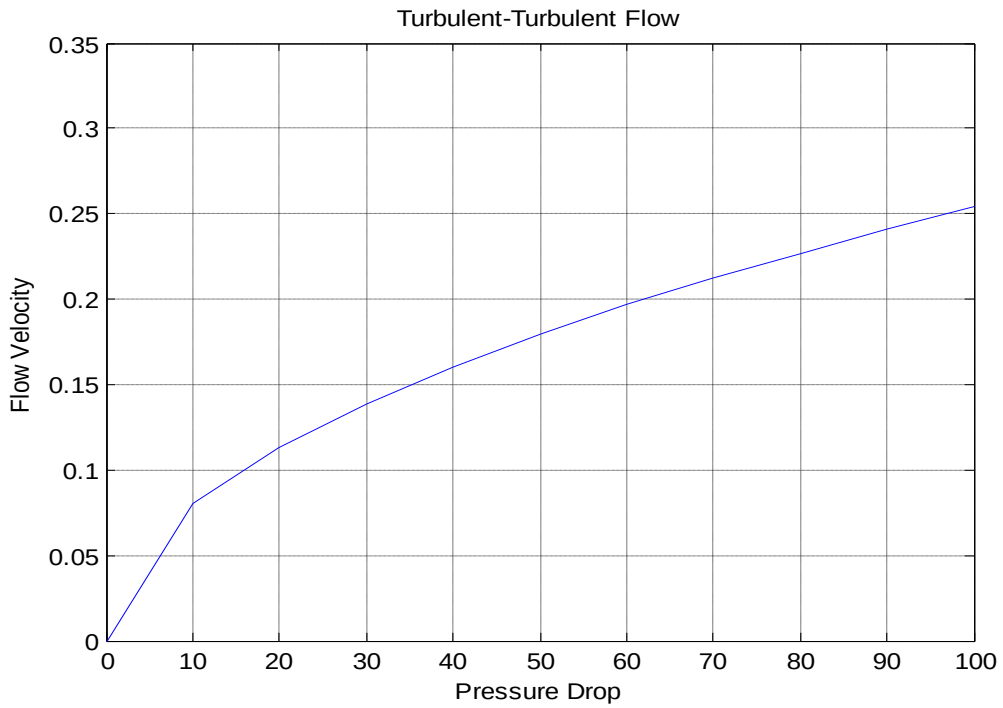

Figure 8Flow Velocity against Pressure for Turbulent-Turbulent Flow

From Figure 8, the flow velocity increases at a constant rate from a pressure drop of zero to a peak value, at a pressure drop of 10 mmHg and then increases slowly as the pressure drop increases. This is as a result of the fact that the turbulence decreases flow at any given pressure. Also turbulence increases the pressure required for a given flow. This happens because turbulence increases the loss of energy in the form of friction which in effect increases the energy required to drive blood flow.

Applications

From the laminar-laminar flow model (Equation 4), the velocity of the blood flow around the clot is directly proportional to the pressure gradient $(p_1 - p_2)$. This means that if the pressure at one side of the blood vessel increases and the pressure at the other side decreases, then the pressure gradient increases. Hence the velocity of blood flow in the vessel also increases. Also the velocity of blood flow is inversely proportional to the viscosity of the blood as can be seen from the expression $U_L = A_1/(8\pi\eta l_1)$. Thus, as the viscosity increases, the denominator of Equation (4) increases and therefore the velocity of the blood flow decreases and this implies that the pressure must be increased to maintain the volume of blood flowing in the vessel. The primary determinants of blood viscosity are hematocrit, red blood cell deformability, red blood cell aggregation, and plasma viscosity. Plasma's viscosity is determined by water-content and macromolecular components, so the major factors that affect blood viscosity are the plasma protein concentration and types of proteins in the plasma (Baskurt and Meiselman, 2003). An increase in viscosity means that the resistance to flow has increased, so a higher pressure would be needed to move the same amount of blood through the arteries and veins. This gives rise to what is known as hyperviscosity. The factors that primarily determine the work of the heart include systolic blood pressure, blood viscosity, and the volume of blood the myocardium has to pump

(Levy and Schiffrin, 2008). The relationship between blood pressure and viscosity as given by Equation (4) is such that, given a constant systolic blood pressure, if blood viscosity increases, then the total peripheral resistance (TPR) will necessarily increase, thereby reducing blood flow. Conversely, when viscosity decreases, blood flow and perfusion will increase. Because of the dependence of systemic arterial blood pressure on cardiac output and TPR, if blood viscosity and TPR rise, systolic blood pressure must then increase for cardiac output to be maintained. Consequently, blood viscosity has been established as a major determinant of the work of the heart and tissue perfusion (Levy and Schiffrin, 2008). Since increased viscosity requires a higher blood pressure to ensure the same circulating volume of blood, both the burden on the heart and the forces acting on the vessel wall are directly modulated by changes in blood viscosity. According to Zarkovic and Kwaan (2003), when viscosity is increased and blood flow is reduced it may possibly result in a number of clinical manifestations. These include lethargy, headache, dehydration, and loss of vision.According to the laminar-turbulent flow model, Equation (7), the velocity of the blood flow around the clot, v_2 is directly proportional to the square root of the pressure gradient. It shows that as the pressure gradient increases the velocity of blood flow in the vessel will increase by only a small amount. Also, Equation (7) shows clearly that as the area of the flow channel along the clot (A_2) reduces at a constant pressure drop, the velocity of the flowing blood also reduces in value. Thus, the pressure gradient must be increased to account for the loss in velocity and so blood flow reduces. This implies that the source of pressure, the heart, must work extra hard in order to increase the blood pressure significantly so as to maintain the volume of blood flowing through the blood vessel. That is exactly why the blood pressure increases when there is blood clot in the artery of the body and this can lead to what is known as hypertension (Lawlor and Smith, 2005).Moreover, from the turbulent-turbulent flow model, that is Equation (9), the velocity of the blood flow around the clot, v_2 is directly proportional to the square root of the pressure gradient. It indicates that as the pressure gradient increases, the velocity of blood flow in the vessel will increase by only a small amount. In this situation, the heart will work and pump harder resulting in an accelerated heartbeat. The elevated cardiac output due to the blood clot in the artery or vein causes turbulence which generates sound waves around the clot that can be heard with a stethoscope. Because higher velocities enhance turbulence, the sound (murmurs) increases as flow increases and may also lead to anemia (Jules, 1999). Abnormalities in the functioning of the heart can also mean that due to turbulence, a speck of the blood clot has moved from the leg and travelled to other parts of the body such as the lungs. This causes sharp, stabbing chest pains which intensify with deep breathing or even cause death. In cases where turbulence around the blood clot is putting substantial pressure on the blood vessel, internal rupture may occur, which may lead to internal bleeding.

Conclusions

Mathematical models for the three flow regimes: laminar-laminar, laminar-turbulent, and turbulent-turbulent flows, have been discussed and the graph of the various models have also been sketched. For the laminar-laminar flow, the flow velocity is directly proportional to the pressure drop. During the laminar-turbulent flow, it was observed that the flow velocity rises slowly as the pressure drop increases. Moreover, in the case of turbulent-turbulent flow the flow velocity rises sharply from an initial value of 0 to a peak of $0.19 m/s$ and continues slowly as the pressure drop increases. The models show that there is a relationship between blood flow velocity and the blood pressure in an artery that is partly blocked by a clot. These models can be very useful in many physiological phenomena of the circulatory system such as the regulation of blood flow in the system or blood flow in a vessel that has become narrowed. Furthermore, if blood flow is laminar then the rate of clot dissolution is directly

proportional to the square of the blood velocity and if blood flow is turbulent, then the rate of clot dissolution is directly proportional to the third power of the blood velocity. These medical models can help medical practitionersand clinicians in improving the quality of life of people and bettermanage clinicallyblood clot diseases.

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