JOINT AND CONDITIONAL R-NORM INFORMATION MEASURE

***Dr Meenakshi Darolia**

The present paper depicts the joint and conditional probability distribution of two random variables ξ and η having probability distributors P and Q over the Sets $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_m\}$respectively. Then the R-norm information of the random variables is denoted by $H_R(\xi) = H_R(P)$ and $H_R(\eta) = H_R(Q)$, where

$$p_i = P(\xi = x_i), i = 1, 2, ..., n \ , \ p_i = P(\eta = y_j), \ j = 1, 2, ..., m$$

are the probabilities of the possible values of the random variables. Similarly, we consider a two-dimensional discrete random variable (ξ,η) with joint probability distribution $\pi = (\pi_{11,} \pi_{12,} ... \pi_{1n})$,

where $\pi_{ij} = P(\xi = x_{i,} \eta = y_j)$, $\mathrm{i} = 1,2.....,\mathrm{n}, \ \mathrm{j} = 1, 2....,\mathrm{m}$ is the joint probability for the values $(x_{i,} y_j)$of (ξ, η). We shall denote conditional probabilities by p_{ij} and q_{ij} such that $\pi_{ij} = p_{ij}\, q_j = q_{ji}\, p_i$ And $p_i = \sum_{j=1}^{m} \pi_{ij} \ and \ q_i = \sum_{j=1}^{n} \pi_{ij}$.

***Assistant Professor, Isharjyot Degree College Pehova**

DEFINITION: The joint R-norm information measure for $R \in R^{+}$ and is given by

$$H_R(\xi,\eta) = \frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}\pi_{ij}{}^{R}\right\}^{\frac{1}{R}}\right] \qquad (1.1)$$

Proposition 1: $H_R(\xi,\eta)$ is symmetric in ξ and η.

Proof: The joint R-norm information measure is defined by

$$H_R(\xi,\eta) = \frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}\pi_{ij}{}^{R}\right\}^{\frac{1}{R}}\right]$$

$$= \frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}P^{R}(\xi = x_i, \eta = y_j)\right\}^{\frac{1}{R}}\right]$$

$$= \frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}P^{R}(\xi = x_i)P^{R}(\eta = y_i)\right\}^{\frac{1}{R}}\right]$$

$$= \frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}P^{R}(\eta = y_i)P^{R}(\xi = x_i)\right\}^{\frac{1}{R}}\right]$$

$$= \frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}P^{R}(\eta = y_i, \xi = x_j)\right\}^{\frac{1}{R}}\right]$$

$$= \frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{n}\sum_{j=1}^{m}\pi_{ji}{}^{R}\right\}^{\frac{1}{R}}\right] = H_R(\eta,\xi)$$

This implies that $H_R(\xi,\eta)$ is symmetric in ξ,η .

Proposition 2: If ξ and η are stochastically independent. Then the following holds

$$H_R(\xi,\eta) = H_R(\xi) + H_R(\eta) - \frac{R-1}{R} H_R(\xi) H_R(\eta) \qquad (1.2)$$

Proof: Since the joint R-norm information measure for $R \in R^+$ and is given by

$$H_R(\xi,\eta) = \frac{R}{R-1}\left[1 - \left\{\sum_{i=1}^{n} \sum_{j=1}^{m} \pi_{ij}{}^{R}\right\}^{\frac{1}{R}}\right]$$

$$= \frac{R}{R-1}\left[1 - \left\{\sum_{i=1}^{n} \sum_{j=1}^{m} P^R\left(\xi = x_i, \eta = y_j\right)\right\}^{\frac{1}{R}}\right]$$

$$= \frac{R}{R-1}\left[1 - \left\{\sum_{i=1}^{n} \sum_{j=1}^{m} P^R\left(\xi = x_i\right) P^R(\eta = y_i)\right\}^{\frac{1}{R}}\right]$$

$$= \frac{R}{R-1}\left[1 - \left\{\sum_{i=1}^{n} \sum_{j=1}^{m} P^R\left(\xi = x_i\right) P^R(\eta = y_i)\right\}^{\frac{1}{R}}\right] \qquad (1.3)$$

Since ξ and η are stochastically independent, thus (1.3) becomes

$$H_R(\xi,\eta) = \frac{R}{R-1}\left[1 - \left[\sum_{i=1}^{n} P^R(\xi = x_i)\right]^{\frac{1}{R}} . \left[\sum_{j=1}^{m} P^R(\eta = y_j)\right]^{\frac{1}{R}}\right]$$

$$= \frac{R}{R-1} - \frac{R}{R-1}\left[\left(1 - \frac{R-1}{R} H_R(\xi)\right)\left(1 - \frac{R-1}{R} H_R(\eta)\right)\right]$$

$$= H_R(\xi) + H_R(\eta) - \frac{R-1}{R} H_R(\xi) H_R(\eta)$$ Thus finally

$$H_R(\xi,\eta) = H_R(\xi) + H_R(\eta) - \frac{R-1}{R} H_R(\xi) H_R(\eta) \qquad (1.4)$$

In the limiting case R→1 we find the additive form of Shannon's information measure for independent random variables.i.e. when R→1 in (1.4), then we get

$$H_R(\xi,\eta) = H_R(\xi) + H_R(\eta) - \frac{1-1}{1} H_R(\xi) H_R(\eta), \quad \Rightarrow H_R(\xi,\eta) = H_R(\xi) + H_R(\eta)$$

To construct a conditional R-norm information measure we can use a direct and an indirect method. The indirect method leads to next definition

DEFINITION: The average subtractive conditional R-norm information of η given ξ for $R \in R^+$ and is defined as

$$^{\delta}H_R(\eta/\xi) = H_R(\xi,\eta) - H_R(\xi)$$

$$= \frac{R}{R-1}\left[1 - \left\{\sum_{i=1}^{n}\sum_{j=1}^{m} \pi_{ij}{}^{R}\right\}^{\frac{1}{R}}\right] - \frac{R}{R-1}\left[1 - \left\{\sum_{i=1}^{n} p_i{}^{R}\right\}^{\frac{1}{R}}\right] \quad (1.4)$$

$$= \frac{R}{R-1}\left[\left\{\sum_{i=1}^{n} p_i{}^{R}\right\}^{\frac{1}{R}} - \left\{\sum_{i=1}^{n}\sum_{j=1}^{m} \pi_{ij}{}^{R}\right\}^{\frac{1}{R}}\right] \quad (1.5)$$

$$= \frac{R}{R-1}\left[1 - \left\{\sum_{i=1}^{n}\sum_{j=1}^{m} \pi_{ij}{}^{R}\right\}^{\frac{1}{R}}\right] - \frac{R}{R-1}\left[1 - \left\{\sum_{i=1}^{n} p_i{}^{R}\right\}^{\frac{1}{R}}\right] \quad (1.6)$$

$$= H_R(\eta/\xi) + H_R(\xi)$$

Thus $H_R(\xi,\eta) = H_R(\eta/\xi) + H_R(\xi)$

A direct way to construct a conditional R-norm information is the following.

DEFINITION: The average conditional R-norm information of η given

ξ is for $R \in R^+$ defined as

$$ {}^{*}H_R(\eta/\xi) = \frac{R}{R-1}\left[1-\sum_{i=1}^{n} p_i\left\{\sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right] \quad (1.7)$$

Or alternatively

$$ {}^{**}H_R(\eta/\xi) = \frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{n} p_i\sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right] \quad (1.8)$$

The two conditional measure given in (1.7) and (1.8) differ by the way the probabilities p_i are incorporated. The expression (1.7) is a true mathematical expression overξ, whereas the expression (1.8) is not.

Theorem: If ξ and η are statistically independent random variables then for R€R$^+$

(1) $${}^{\delta}H_R(\eta/\xi) = \frac{R}{R-1}\left[\left\{\sum_{i=1}^{n} p_i{}^{R}\right\}^{\frac{1}{R}} - \left\{\sum_{i=11}^{n} p_i{}^{R}\right\}^{\frac{1}{R}} \cdot \left\{\sum_{j=}^{m} q_j{}^{R}\right\}^{\frac{1}{R}}\right]$$

(2) $${}^{\delta}H_R(\eta/\xi) = H_R(\xi,\eta) - H_R(\xi) = H_R(\eta) - \frac{R-1}{R}H_R(\xi)H_R(\eta)$$

(3) $${}^{*}H_R(\eta/\xi) = H_R(\eta)$$

(4) $^{**}H_R(\eta/\xi) = H_R(\eta)$

Proof: (I) Since the average subtractive conditional R-norm information of η given ξ is for $R \in R^+$ defined as

$$^{\delta}H_R(\eta/\xi) = \frac{R}{R-1}\left[\left\{\sum_{i=1}^{n} p_i^{\,R}\right\}^{\frac{1}{R}} - \left\{\sum_{i=1}^{n}\sum_{j=1}^{m} \pi_{ij}^{\,R}\right\}^{\frac{1}{R}}\right] \tag{1.9}$$

Substitute $\pi_{ij} = p_{ij} q_j$ in (1.9), we get

$$^{\delta}H_R(\eta/\xi) = \frac{R}{R-1}\left[\left\{\sum_{i=1}^{n} p_i^{\,R}\right\}^{\frac{1}{R}} - \left\{\sum_{i=1}^{n}\sum_{j=1}^{m} (p_{ij} q_j)^R\right\}^{\frac{1}{R}}\right] \tag{1.10}$$

Since ξ and η are stochastically independent. Thus (1.10) becomes

$$^{\delta}H_R(\eta/\xi) = \frac{R}{R-1}\left[\left\{\sum_{i=1}^{n} p_i^{\,R}\right\}^{\frac{1}{R}} - \left\{\sum_{i=1}^{n} p_i^{\,R}\right\}^{\frac{1}{R}} \cdot \left\{\sum_{j=1}^{m} q_j^{\,R}\right\}^{\frac{1}{R}}\right]$$

(II) Since we know that if ξ and η are stochastically independent. Then the following holds

$$H_R(\xi,\eta) = H_R(\xi) + H_R(\eta) - \frac{R-1}{R} H_R(\xi) H_R(\eta)$$

$$\Rightarrow H_R(\xi,\eta) - H_R(\xi) = H_R(\eta) - \frac{R-1}{R} H_R(\xi) H_R(\eta) \tag{1.11}$$

And we know

$$^{\delta}H_R(\eta/\xi) = H_R(\xi,\eta) - H_R(\xi) \tag{1.12}$$

Using (1.12) in (1.11), we get

$$^{\delta}H_R(\eta/\xi) = H_R(\xi,\eta) - H_R(\xi) = H_R(\eta) - \frac{R-1}{R}H_R(\xi)H_R(\eta)$$

(III) Since the average conditional R-norm information of η given ξ for $R \in R^{+}$ and is defined as

$$^{*}H_R(\eta/\xi) = \frac{R}{R-1}\left[1 - \sum_{i=1}^{n} p_i\left\{\sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right] \tag{1.13}$$

Substitute $q_{ji} = q_j$ in (1.13), we get

$$^{*}H_R(\eta/\xi) = \frac{R}{R-1}\left[1 - \sum_{i=1}^{n} p_i\left\{\sum_{j=1}^{m} q_j{}^{R}\right\}^{\frac{1}{R}}\right]$$

$$= \frac{R}{R-1}\left[1 - \left\{\sum_{j=1}^{m} q_j{}^{R}\right\}^{\frac{1}{R}}\right] = H_R(\eta) \qquad \left[\Theta\left[\sum_{i=1}^{n} p_i\right] = 1\right]$$

Hence $^{*}H_R(\eta/\xi) = H_R(\eta)$ (1.14)

(IV) Since the average conditional R-norm information of η given ξ for $R \in R^{+}$

and is defined as

$$^{**}H_R(\eta/\xi) = \frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right] \qquad (1.15)$$

Substitute $q_{ji} = q_j$ in (1.15), we get

$$^{**}H_R(\eta/\xi) = \frac{R}{R-1}\left[1-\left\{\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_j{}^{R}\right\}^{\frac{1}{R}}\right]$$

$$= \frac{R}{R-1}\left[1-\left\{\sum_{j=1}^{m} q_j{}^{R}\right\}^{\frac{1}{R}}\right] = H_R(\eta) \qquad \left[\Theta\left[\sum_{i=1}^{n} p_i\right] = 1\right]$$

Hence $^{**}H_R(\eta/\xi) = H_R(\eta)$

From this theorem we may conclude that the measure $^{\delta}H_R(\eta/\xi)$, which is obtained by the formal difference between the joint and the marginal information measure, does not satisfy requirement (I). Therefore it is less attractive than the two other measure. In the next theorem we consider requirement (II), for the conditional information measures $^{*}H_R(\eta/\xi)$ and $^{**}H_R(\eta/\xi)$.

Theorem: If ξ and η are discrete random variables then for $R \in R^{+}$ then the following results hold.

(I) $^{*}H_R(\eta/\xi) \le H_R(\eta)$ (II) $^{**}H_R(\eta/\xi) \le H_R(\eta)$

(III) $^{**}H_R(\eta/\xi) \le {}^{*}H_R(\eta/\xi)$ (IV) $^{**}H_R(\eta/\xi) \le {}^{*}H_R(\eta/\xi) \le H_R(\eta)$

The equality signs holds if ξ and η are independent. **Proof:**

(I) Here we consider two cases: Cases I: when R < 1

We know by [4] that for $R > 1$.

$$\left[\sum_{j=1}^{m}\left\{\sum_{i=1}^{n} x_{ij}\right\}^{R}\right]^{\frac{1}{R}} \le \left[\sum_{i=1}^{n}\left\{\sum_{j=1}^{m} x_{ij}^{R}\right\}^{\frac{1}{R}}\right] \tag{1.16}$$

Setting $x_{ij} = \pi_{ij} \ge 0$ in (1.16), we have

$$\left[\sum_{j=1}^{m}\left\{\sum_{i=1}^{n} \pi_{ij}\right\}^{R}\right]^{\frac{1}{R}} \le \left[\sum_{i=1}^{n}\left\{\sum_{j=1}^{m} \pi_{ij}^{R}\right\}^{\frac{1}{R}}\right] \tag{1.17}$$

Since $q_i = \sum_{j=1}^{m} \pi_{ij}$ and $\pi_{ij} = p_i q_{ji}$ (1.18)

Using (1.18) in (1.17), we get

$$\left[\sum_{j=1}^{m} q_j^{R}\right]^{\frac{1}{R}} \le \left[\left\{\sum_{i=1}^{n} \sum_{j=1}^{m} (q_{ji} p_i)^{R}\right\}^{\frac{1}{R}}\right] \tag{1.19}$$

It can be written as

$$\left[\sum_{j=1}^{m} q_j^{\,R}\right]^{\frac{1}{R}} \leq \left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}^{\,R}\right\}^{\frac{1}{R}}\right]$$

$$-\left[\sum_{j=1}^{m} q_j^{\,R}\right]^{\frac{1}{R}} \geq -\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}^{\,R}\right\}^{\frac{1}{R}}\right]$$

$$1-\left[\sum_{j=1}^{m} q_j^{\,R}\right]^{\frac{1}{R}} \geq 1-\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}^{\,R}\right\}^{\frac{1}{R}}\right] \qquad (1.20)$$

We know $\frac{R}{R-1} > 0$ if $R > 1$

Multiplying both sides of (1.20) by $\frac{R}{R-1}$, we get

$$\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} q_j^{\,R}\right]^{\frac{1}{R}}\right] \geq \frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}^{\,R}\right\}^{\frac{1}{R}}\right]\right] \qquad (1.21)$$

But $\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}^{\,R}\right\}^{\frac{1}{R}}\right]\right] =^{+} H_R(\eta/\xi)$ and

$\frac{R}{R-1}\left[1-\left[\sum_{j=1}^{m} q_j^{\,R}\right]^{\frac{1}{R}}\right] = H_R(\eta)$ Thus (1.21) becomes

$$^{*}H_R(\eta/\xi) \leq H_R(\eta) \quad \text{for } R > 1 \qquad (1.22)$$

Cases II: when 0 <R < 1

We know by [4] that for 0 < R < 1

$$\left[\sum_{j=1}^{m}\left\{\sum_{i=1}^{n}x_{ij}\right\}^{R}\right]^{\frac{1}{R}} \geq \left[\sum_{i=1}^{n}\left\{\sum_{j=1}^{m}x_{ij}^{R}\right\}^{\frac{1}{R}}\right] \qquad (1.23)$$

Setting $x_{ij} = \pi_{ij} \geq 0$ in (1.23), we have

$$\left[\sum_{j=1}^{m}\left\{\sum_{i=1}^{n}\pi_{ij}\right\}^{R}\right]^{\frac{1}{R}} \geq \left[\sum_{i=1}^{n}\left\{\sum_{j=1}^{m}\pi_{ij}^{R}\right\}^{\frac{1}{R}}\right] \qquad (1.24)$$

Since $q_i = \sum_{j=1}^{m}\pi_{ij}$

Thus (1.24) becomes

$$\left[\sum_{j=1}^{m} q_j^{\,R}\right]^{\frac{1}{R}} \geq \left[\sum_{i=1}^{n}\left\{\sum_{j=1}^{m}\left(q_{ji}p_i\right)^{R}\right\}^{\frac{1}{R}}\right]$$

$$-\left[\sum_{j=1}^{m} q_j^{\,R}\right]^{\frac{1}{R}} \leq -\left[\sum_{i=1}^{n}\left\{\sum_{j=1}^{m}\left(q_{ji}p_i\right)^{R}\right\}^{\frac{1}{R}}\right]$$

$$\Rightarrow 1-\left[\sum_{j=1}^{m} q_j{}^R\right]^{\frac{1}{R}} \le 1-\left[\sum_{i=1}^{n}\left\{\sum_{j=1}^{m}\left(q_{ji}\,p_i\right)^R\right\}^{\frac{1}{R}}\right] \qquad (1.25)$$

We know $\frac{R}{R-1}<0$ if $0 < R < 1$

Multiplying both sides of (1.25) by $\frac{R}{R-1}$, we get

$$\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} q_j{}^R\right]^{\frac{1}{R}}\right] \ge \frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i\left\{\sum_{j=1}^{m} q_{ji}{}^R\right\}^{\frac{1}{R}}\right]\right] \qquad (1.26)$$

But $$\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i\left\{\sum_{j=1}^{m} q_{ji}{}^R\right\}^{\frac{1}{R}}\right]\right] = H_R(\eta/\xi) \quad \text{and}$$

$$\frac{R}{R-1}\left[1-\left[\sum_{j=1}^{m} q_j{}^R\right]^{\frac{1}{R}}\right] = H_R(\eta)$$

Thus (1.26) becomes

$$\Rightarrow^{*} H_R(\eta/\xi) \le H_R(\eta) \quad \text{for } 0 < R < 1 . \qquad (1.27)$$

Thus from (1.22) and (1.27), we get

$${}^{*}H_R(\eta/\xi) \le H_R(\eta) \quad \text{for R € R}^{+}$$

(II) H ere we consider two cases

Cases I: when R >1

From Jensen's inequality for $R > 1$,we find

$$\sum_{i=1}^{n} p_i q_{ji}{}^{R} \geq \left[\sum_{i=1}^{n} p_i q_{ji}\right]^{R} = q_j^R \qquad (1.28)$$

After summation over j and raising both sides of (1.28) by power $\frac{1}{R}$, we have

$$\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}} \geq \left[\sum_{j=1}^{m} q_j{}^{R}\right]^{\frac{1}{R}}$$

$$-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}} \leq -\left[\sum_{j=1}^{m} q_j{}^{R}\right]^{\frac{1}{R}}$$

$$1-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}} \leq 1-\left[\sum_{j=1}^{m} q_j{}^{R}\right]^{\frac{1}{R}} \qquad (1.29)$$

Using $\frac{R}{R-1} > 0$ as R > 1, Thus (1.29) becomes

$$\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}}\right] \leq \frac{R}{R-1}\left[1-\left[\sum_{jj=1}^{m} q_j{}^{R}\right]^{\frac{1}{R}}\right] \qquad (1.30)$$

But $$\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}}\right] =^{++} H_R(\eta/\xi)$$

And $$\frac{R}{R-1}\left[1-\left[\sum_{jj=1}^{m} q_j{}^{R}\right]^{\frac{1}{R}}\right] = H_R(\eta)$$

Thus (1.30) becomes

$$^{**}H_R(\eta/\xi) \le H_R(\eta) \quad \text{for } R > 1 \tag{1.31}$$

Case II: when 0 < R <1

From Jensen's inequality for 0 < R < 1 we find

$$\sum_{i=1}^{n} p_i q_{ji}{}^{R} \le \left[\sum_{i=1}^{n} p_i q_{ji}\right]^{R} = q_j^{R} \tag{1.32}$$

After summation over j and raising both sides of (1.32) by power $\frac{1}{R}$, we have

$$\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}} \le \left[\sum_{j=1}^{m} q_j{}^{R}\right]^{\frac{1}{R}}$$

$$-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}} \ge -\left[\sum_{j=1}^{m} q_j{}^{R}\right]^{\frac{1}{R}}$$

$$1-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}} \ge 1-\left[\sum_{j=1}^{m} q_j{}^{R}\right]^{\frac{1}{R}} \tag{1.33}$$

Using $\frac{R}{R-1} < 0$ as 0 < R <1, Thus (1.33) becomes

$$\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}}\right] \leq \frac{R}{R-1}\left[1-\left[\sum_{jj=1}^{m} q_j{}^{R}\right]^{\frac{1}{R}}\right] \tag{1.34}$$

But $$\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}}\right] =^{++} H_R(\eta/\xi)$$

And $$\frac{R}{R-1}\left[1-\left[\sum_{jj=1}^{m} q_j{}^{R}\right]^{\frac{1}{R}}\right] = H_R(\eta)$$

Thus (1.34) becomes

$$^{**}H_R(\eta/\xi) \leq H_R(\eta) \quad \text{for} \quad 0 < R < 1 \tag{1.35}$$

Thus from (1.31) and (1.35), we get

$$^{**}H_R(\eta/\xi) \leq H_R(\eta) \quad \text{for} \quad R \text{ € } R^{+}$$

(III) Here we consider two cases:

Cases I: when R >1

We know from Jensen's inequality

$$\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right] \leq \left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}} \quad \text{for } R>1 \tag{1.36}$$

$$-\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right] \geq -\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}}$$

$$1-\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right] \geq 1-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}} \qquad (1.37)$$

Using $\frac{R}{R-1} > 0$ if R >1, then (1.37) becomes

$$\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right]\right] \geq \frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}}\right] \qquad (1.38)$$

But $$\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right]\right] =^{+} H_R(\eta/\xi)$$

And $$\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}}\right] =^{++} H_R(\eta/\xi)$$

Thus (1.38) becomes $^{**}H_R(\eta/\xi) \leq {}^{*}H_R(\eta/\xi)$ for R > 1 (1.39) **Case II: when 0 < R < 1 We know from Jensen's inequality**

$$\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right] \geq \left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}} \quad \text{for } 0 < R < 1 \qquad (1.40)$$

$$-\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right] \le -\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}}$$

$$1-\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right] \le 1-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}} \qquad (1.41)$$

Using $\frac{R}{R-1} < 0$ if 0 < R < 1, then (3.41) becomes

$$\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right]\right] \ge \frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}}\right] \qquad (1.42)$$

But $\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \left\{\sum_{j=1}^{m} q_{ji}{}^{R}\right\}^{\frac{1}{R}}\right]\right] =^{+} H_R(\eta/\xi)$

And $\frac{R}{R-1}\left[1-\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}{}^{R}\right]^{\frac{1}{R}}\right] =^{++} H_R(\eta/\xi)$ Thus (1.42) becomes

$$^{**}H_R(\eta/\xi) \le {}^{*}H_R(\eta/\xi) \qquad \text{for } 0 < R < 1 \qquad (1.43)$$

Thus from (1.39) and (1.43), we get $^{**}H_R(\eta/\xi) \le {}^{*}H_R(\eta/\xi)$ for R € R^+

(IV) From (I) and (III), we have $^{*}H_R(\eta/\xi) \le H_R(\eta)$ And

$^{**}H_R(\eta/\xi) \le {}^{*}H_R(\eta/\xi)$ Thus finally we find

$$^{**}H_R(\eta/\xi) \le {}^{*}H_R(\eta/\xi) \le H_R(\eta) \qquad (1.44)$$

HENCE PROVED

REFRENCES

[1]. ACZEL, J. AND DARCOZY, Z. (1975), "On Measure of Information and their Characterizations," Academic Press, New York.

[2]. ARIMOTO , S. (1971), Information theoretical considerations on problems, Inform. Contr. 19, 181-194.

[3]. BECKENBACH, E.F. AND BELLMAN, R.(1971),'Inequalities', Springer- Verlag, Berlin.

[4]. BOEKEE, D.E. AND VAR DER LUBEE, J. C.A. (1979), Some aspects of error bounds in features selection, Pattern Recognition 11, 353-360.

[5]. CAMPBELL ,L.L (1965), "A Coding Theorem and Renyi's Entropy", Information and Control, 23, 423-429.

[6]. DAROCZY, Z. (1970), Generalized information function, Inform. Contr.16, 36-51.

[7]. D. E. BOEKKE AND J. C. A. VAN DER LUBBE ,"R-Norm Information Measure", Information and Control. 45, 136-155(1980

[8]. GYORFI ,L. AND NEMETZ, T. (1975), "On the dissimilarity of probability measure, in Proceedings Colloq. on Information Theory, Keshtely, Hungary."

[9]. HARDY, G. H., LITTLEWOOD, J. E., AND POLYA, G.(1973), "Inequalities Cambridge Univ. Press, London /New York

[10]. HAVDRA, J. AND CHARVAT, F. (1967). Quantification method of Classification processes, Concept of structural α- entropy, Kybernetika 3, 30-35.

[11]. Nath, P.(1975), On a Coding Theorem Connected with Renyi's Entropy Information and Control 29, 234-242.

[12]. O.SHISHA, Inequalities,Academic Press, New York.

[13]. RENVI, A. (1961),On measure of entropy and information, in proceeding, Fourth Berkeley Symp. Math. Statist. Probability No-1, pp. 547-561.

[14]. R. P. Singh (2008), Some Coding Theorem for weighted Entropy of order α.. Journal of Pure and Applied Mathematics Science : New Delhi.