ATTRACTIVITY OF A HIGHER ORDER NONLINEAR DIFFERENCE EQUATION

Guangfeng Liu

School of Zhangjiagang Jiangsu University of Science and Technology, Zhangjiagang, Jiangsu 215600,P.R. China

Abstract: In this paper, the global asymptotic behavior of the following rational difference equation

$$y_{n+1}=\frac{r+py_n+y_{n-k}}{qy_n+y_{n-k}},n=0,1,2,\cdots,$$

is investigated, where the parameters $p,q,r\in(0,\infty)$, and the initial conditions $y_{-k},\cdots,y_{-1}$ are positive real numbers and y_0 is a positive real number.

Keywords: Locally stability; Rational difference equation; Global attractivity; Global asymptotical stability.

1. Introduction

Consider the following rational difference equation

$$y_{n+1}=\frac{r+py_n+y_{n-k}}{qy_n+y_{n-k}},n=0,1,2,\cdots, \tag{1.1}$$

where the parameters p,q,r are positive real numbers and the initial conditions $y_{-k},\cdots,y_{-1}\in[0,\infty)$ and $y_0\in(0,\infty)$.

For the equation

$$y_{n+1}=\frac{r+py_n+y_{n-1}}{qy_n+y_{n-1}},n=0,1,2,\cdots, \tag{1.2}$$

M. R. S. Kulenovic and G. Ladas in their monograph $[2,P_{181}]$ presented the following problem.

Conjecture 10.5.2

Assume that $p,q,r\in(0,\infty)$.Then the following statements are true for Eq.(1.1).

(a) Local Stability of the positive equilibrium implies global stability.

(b) When Eq.(1.1) has no prime period-two solution, the equilibrium $\bar{y}$ is globally asymptotically stable.

(c) When Eq.(1.1) possesses a period-two solution, the equilibrium $\bar{y}$ is a saddle point.

(d) The period-two solution of Eq.(1.1),when it exists, is locally asymptotically stable, but not globally.

Motivated by the above conjecture, more generally, our main aim in this paper is to investigate the local stability and the global attractivity of the equilibrium $\bar{y}$ of Eq.(1.1).

Here, for the sake of convenience, we now give some corresponding definitions, also review some known lemmas which will be useful in the sequel.

Consider the different equation

$$x_{n+1} = F(x_{n,} x_{n-1}, \cdots, x_{n-k}), n = 0,1,\cdots, \tag{1.3}$$

where is a positive integer, and the function $F(u_0, u_1, \cdots, u_k)$ has continuous partial derivatives.

A point $\bar{x}$ is called an equilibrium of Eq.(1.3), if

$$\bar{x} = F(\bar{x}, \bar{x}, \cdots, \bar{x}).$$

That is $x_n = \bar{x}$ for $n > 0$ is a solution of Eq.(1.3) or equivalently, $\bar{x}$ is a fixed point of F.

The linearized equation associated with Eq.(1.3) about the equilibrium point $\bar{x}$ is

$$x_{n+1} = \sum_{i=1}^{k} \frac{\partial F}{\partial u_i}(\bar{x}, \bar{x}, \cdots, \bar{x}) x_{n-i} \quad n = 0,1,\cdots.$$

Its characteristic equation is

$$\lambda^{n+1} = \sum_{i=1}^{k} \frac{\partial F}{\partial u_i}(\bar{x}, \bar{x}, \cdots, \bar{x}) \lambda^{n-i} \quad n = 0,1,\cdots.$$

In the following, we present some Lemmas, which will be used in the sequel.

Lemma A[1]. Assume that $\alpha, \beta \in \{0,1,2,\cdots\}$. Then

$$|\alpha| + |\beta| < 1 \tag{1.4}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} - \alpha x_n + \beta x_{n-k} = 0, n = 0,1,2,\cdots, \tag{1.5}$$

Suppose in addition that one of the following two cases holds:

(a) k is odd and $\beta < 0$;
(b) k is even and $\alpha\beta < 0$;
Then (1.4) is also a necessary condition for the asymptotic stability of Eq.(1.5).

Lemma B[1]. Let $H \in [[0,\infty),(0,\infty)]$ be a nonincreasing function and let $\bar{x}$ denote the (unique) fixed point of H. Then the following statements are equivalent:

(a) $\bar{x}$ is the only fixed point of $H^2 \in (0,\infty)$;

(b) $\bar{x}$ is a global attractor of all positive solution of the equation

$$x_{n+1} = H(x_n), n = 0,1,2,\cdots, \tag{1.6}$$

with $x_0 \in [0,\infty)$.

Lemma C[1]. Consider the difference equation

$$x_{n+1} = x_n f(x_, x_{n-k_1}, \cdots, x_{n-k_r}), n = 0,1,2,\cdots, \tag{1.7}$$

where $k_i (i = 1,2,\cdots r)$ are positive integers. Denote by k the maximum of $k_1,\cdots,k_r$. Also, assume that the function f satisfies the following hypotheses:

(H1) $f \in C[0,\infty)\times[0,\infty)^r,(0,\infty)]$ and $g \in [[0,\infty)^{r+1},(0,\infty)]$, where $g(u_0,u_1,\cdots,u_r) = u_0 f(u_0,u_1,\cdots,u_r)$ for $u_0 \in (0,\infty)$ and $u_0,\cdots,u_r \in [0,\infty)$, and

$$g(0,u_1,\cdots,u_r) = \lim_{u_0\to 0^+} g(u_0,u_1,\cdots,u_r);$$

(H2) $f(u_0,u_1,\cdots,u_r)$ is nonincreasing in $u_1,\cdots,u_r$;

(H3) The equation $f(x,x,\cdots,x)$ has a unique positive solution $\bar{x}$;

(H4) Either the function $f(u_0,u_1,\cdots,u_r)$ does not depend on u_0 or for every x>0 and $u \ge 0$,

$$[f(x,u,u,\cdots,u) - f(\bar{x},u,u,\cdots,u)](x-\bar{x}) \le 0$$

with

$$[f(x,\bar{x},\bar{x},\cdots,\bar{x}) - f(\bar{x},\bar{x},\bar{x},\cdots,\bar{x})](x-\bar{x}) < 0 \text{ for } x \ne \bar{x}.$$

Now, define a new function:

$$F(x) = \begin{cases} \max_{x\le y\le \bar{x}} G(x,y)\, for 0 \le x \le \bar{x}, \\ \min_{\bar{x}\le y\le x} G(x,y)\, for x > \bar{x}, \end{cases}$$

where

$$G(x,y) = yf(y,x,\cdots,x)f(\bar{x},\bar{x},\cdots,\bar{x},y)[f(\bar{x},x,\cdots,x)]^{k+1}. \tag{1.8}$$

Then

(a) $F \in C[(0,\infty),(0,\infty)]$ and F is nonincreasing in $(0,\infty)$.

(b) Assume that F has no periodic points of prime period two. Then $\bar{x}$ is a global attractor of all positive solutions of Eq.(1.7).\

Lemma D[1]. Let F, H $\in$ C[(0,∞),(0,∞)] be nonincreasing functions in F, H (0,∞) be such that

$$F(\bar{x})=H(\bar{x})=\bar{x},$$

and

$$[H(x)-F(x)](x-\bar{x}) \le 0 \text{ for } x \ge 0.$$

Assume that $\bar{x}$ is the only fixed point of H^2 in $(0,\infty)$. Then $\bar{x}$ is also the only fixed point of F^2 in $(0,\infty)$.

Lemma E[1]. Consider the difference equation

$$y_{n+1} = f(y_n, y_{n-k}), n = 0,1,\cdots, \tag{1.9}$$

where $k \in \{1,2,\cdots\}$, let I=[a,b] be some interval of real numbers and assume that

$$f:[a,b]\times[a,b] \to [a,b]$$

is a continuous function satisfying the following properties:

(a) f(x,y) is nonincreasing in each of its arguments;

(b) If $(m,M) \in [a,b]\times[a,b]$ is a solution of the system

$$f(m,m) = M \quad and \quad f(M,M) = m,$$

then m=M.

Then Eq.(1.9) has a unique positive equilibrium $\bar{y}$ and every solution of Eq.(1.9) converges to $\bar{y}$.

Lemma F[1]. Let $I=[a,b]$ be an invariant interval under a continuous function G(x, y) which is nonincreasing in x for each $y \in I$ and nondecreasing in y for each $x \in I$. Assume that $\bar{y} \in I$ is a unique equilibrium point of the equation

$$y_{n+1} = G(y_n, y_{n-k}), n = 0,1,\cdots. \qquad (*)$$

If the system

$$x = G(y, x) \text{ and } y = G(x, y) \qquad (1.10)$$

Has exactly one solution in I^2, then $\bar{y}$ is a global attractor with basin I^2.

2. Boundness and Persistence

In this section, we will consider the boundedness and persistence of Eq.(1.1). We have the following result.

Lemma 2.1 Assume that $p,q,r \in (0,\infty)$ with $p<q$, then every positive solution of Eq.(1.1) is bounded and persists.

Proof. Let $\{y_n\}_{n=-1}^{\infty}$ be an arbitrary positive solution of Eq.(1.1). Then, it follows from Eq.(1.1) that

$$y_{n+1} = \frac{r + py_n + y_{n-k}}{qy_n + y_{n-k}} = \frac{py_n + \frac{p}{q} y_{n-k}}{qy_n + y_{n-k}} + \frac{r + y_{n-k} - \frac{p}{q} y_{n-k}}{qy_n + y_{n-k}} > \frac{p}{q}, n \geq 0$$

and for $n \geq 2$,

$$y_{n+1} = \frac{r + py_n + y_{n-k}}{qy_n + y_{n-k}} < \frac{py_n + y_{n-k}}{qy_n + y_{n-k}} + \frac{r}{(q+1)\frac{p}{q}} < 1 + \frac{rq}{p(q+1)}.$$

This complete the proof.

3. Local stability and Global stability

In this section, we will consider the global asymptotic stability of Eq.(1.1) where $p,q,r \in (0,\infty)$, the initial conditions

$$y_{-k},\cdots,y_{-1} \in [0,\infty) \quad and \quad y_0 \in (0,\infty). \qquad (3.1)$$

Eq.(1.1) has the unique positive equilibrium $\bar{y}$ given by

$$\bar{y}=\frac{p+1+\sqrt{(p+1)^2+4r(q+1)}}{2(q+1)}.$$

The liberalized equation associated with equation Eq.(1.1) about $\bar{y}$ is

$$Z_{n+1}-\frac{\bar{y}(p-q)-qr}{\bar{y}^2(q+1)^2}Z_n+\frac{\bar{y}(p-q)-r}{\bar{y}^2(q+1)^2}Z_{n-k}=0,n=0,1,2,\cdots,$$

and its characteristic equation is

$$\lambda^{k+1}-\frac{\bar{y}(p-q)-qr}{\bar{y}^2(q+1)^2}\lambda^k+\frac{\bar{y}(p-q)-r}{\bar{y}^2(q+1)^2}=0.$$

From this and by lemma A, we have the following result.

Theorem 3.1 Assume that $p,q,r\in(0,\infty)$ and the initial conditions

$$y_{-k},\cdots,y_{-1}\in[0,\infty) \text{ and } y_0\in(0,\infty). \tag{3.2}$$

Hold. Then the following results are true.

(1) If $(q-p)\bar{y}+qr\geq 0,(p-q)\bar{y}+r\geq 0$ and $r<\bar{y}(q+1)$,
then the unique positive equilibrium $\bar{y}$ of Eq.(1.1) is locally asymptotically stable

(2) If $(q-p)\bar{y}+r<0$ and $2(q-p)\bar{y}+(q-1)r<\bar{y}^2(q+1)^2$,
then the unique positive equilibrium $\bar{y}$ of Eq.(1.1) is locally asymptotically stable

(3) If $(q-p)\bar{y}+qr<0$ and $2(q-p)\bar{y}-(q-1)r<\bar{y}^2(q+1)^2$,
then the unique positive equilibrium $\bar{y}$ of Eq.(1.1) is locally asymptotically stable

(4) If $(q-p)\bar{y}+r<0$ and $2(q-p)\bar{y}-(q-1)r<\bar{y}^2(q+1)^2$,
then the unique positive equilibrium $\bar{y}$ of Eq.(1.1) is unstable

Theorem 3.2 Assume that $p,q,r\in(0,\infty)$. Then the unique positive equilibrium $\bar{y}$ of Eq.(1.1) is a global attractor provided that one of the following conditions is satisfied:

i) p >q;

ii) $1<p\leq q$.

Proof. First, we consider the case i) p > q. Then Eq.(1.1) can be rewritten as follows:

$$y_{n+1} = y_n \frac{p + \frac{r + y_{n-k}}{y_k}}{qy_n + y_{n-k}}.$$

Set

$$f(u_0, u_k) = \frac{p + \frac{r + u_k}{u_0}}{qu_0 + u_k} \quad and \quad g(u_0, u_k) = u_0 f(u_0, u_k)$$

It is easy to verify that the function f satisfy the hypotheses of (H1)-(H4) of Lemma C. Furthermore the function G, which is defined by (1.6) takes the form

$$G(y,z) = \frac{1}{\bar{y}} \frac{r + y + pz}{qz + y} \frac{r+z+p\bar{y}}{q\bar{y}+z} [\frac{r+y+p\bar{y}}{q\bar{y}+y}]^{k-1} .$$

Next, we will construct the function F defined by(1.6). Since

$$\frac{d}{dz}(\frac{r + y + pz}{qz + y} \frac{r + z + p\bar{y}}{q\bar{y} + z}) = \frac{(p-q)y\bar{y} + qr(z-\bar{y}) + r(y-z) + (q-p)z^2}{(qz+y)^2 (q\bar{y}+z)^2}.$$

Case(1) $(p-q)y\bar{y} + qr(z-\bar{y}) + r(y-z) + (q-p)z^2 \geq 0$.

Then the function

$$\frac{pz + r + y}{qz + y}$$

is nondecreasing in z and so

$$\max_{y \leq z \leq \bar{y}} \frac{r + y + pz}{qz + y} \frac{r + z + p\bar{y}}{q\bar{y} + z} = \frac{r + y + p\bar{y}}{q\bar{y} + y} \frac{r + \bar{y} + p\bar{y}}{q\bar{y} + \bar{y}}.$$

Also

$$\min_{\bar{y} \leq z \leq y} \frac{r + y + pz}{qz + y} \frac{r + z + p\bar{y}}{q\bar{y} + z} = \frac{r + y + p\bar{y}}{q\bar{y} + y} \frac{r + \bar{y} + p\bar{y}}{q\bar{y} + \bar{y}}.$$

Hence the function F is given by

$$F(y) = \frac{1}{\bar{y}} \frac{r + \bar{y} + p\bar{y}}{q\bar{y} + \bar{y}} [\frac{r + y + p\bar{y}}{q\bar{y} + y}]^N,$$

In order to apply Lemma D to show that $\bar{y}$ is the only fixed point of F^2 in $(0,\infty)$. Let

$$H(y)=\frac{1}{\bar{y}}\frac{r+\bar{y}+p\bar{y}}{q\bar{y}+\bar{y}}[\frac{r+y+p\bar{y}}{q\bar{y}+y}]^{N},$$

where N=n+k, n is a sufficiently large positive integer. From this we know that the function $H(y)\in C[(0,\infty),(0,\infty)]$ is strictly decreasing in $(0,\infty)$, and $H(\bar{y})=F(\bar{y})=\bar{y}$.

Let L=H(M) and M>0 is the fixed point of $H^{2}(y)$, that is to say, $H^{2}(M)=M$. Then

$$H(L)=\frac{1}{\bar{y}}\frac{r+\bar{y}+p\bar{y}}{q\bar{y}+\bar{y}}[\frac{r+\bar{y}+p\bar{y}}{q\bar{y}+L}]^{N}=M$$

and

$$H(M)=\frac{1}{\bar{y}}\frac{r+\bar{y}+p\bar{y}}{q\bar{y}+\bar{y}}[\frac{r+\bar{y}+p\bar{y}}{q\bar{y}+M}]^{N}=L$$

hence

$$L[\frac{1}{q\bar{y}+L}]^{N}=M[\frac{1}{q\bar{y}+M}]^{N}$$

Let

$$R(x)=x[\frac{1}{q\bar{y}+x}]^{N}\quad \text{for } x\in(0,\infty).$$

Then

$$R^{'}(x)=[\frac{1}{q\bar{y}+x}]^{N}\quad(1-\frac{N}{q\bar{y}}).$$

Since N is a sufficiently large positive integer, then for any $x\in(0,\infty)$, we can obtain a sufficiently large positive integer N, such that

$$(1-\frac{N}{q\bar{y}})\leq 0.$$

Thus the function R(x) is strictly decreasing for $x\in(0,\infty)$ and M=L. So $\bar{y}$ is the unique fixed point of $H^{2}(y)$ in $y\in(0,\infty)$.

For $y\geq 0$ we know that

$$[H(y)-F(y)](y-\bar{y})=\frac{1}{\bar{y}}\frac{r+\bar{y}+p\bar{y}}{q\bar{y}+\bar{y}}[\frac{r+y+p\bar{y}}{q\bar{y}+y}]^{k}[(\frac{r+y+p\bar{y}}{q\bar{y}+y})^{n}-1](y-\bar{y})$$
$$=-(y-\bar{y})^{2}\frac{1}{\bar{y}}\frac{r+\bar{y}+p\bar{y}}{q\bar{y}+\bar{y}}[\frac{r+y+p\bar{y}}{q\bar{y}+y}]^{k}\times(r+y+p\bar{y})^{n-1}+(r+y+p\bar{y})^{n-1}$$
$$+(r+y+p\bar{y})^{n-2}(q\bar{y}+y)+\cdots+(q\bar{y}+y)^{n-1}\leq 0.$$

By Lemma D, $\bar{y}$ is also the only fixed point of F^2 in $(0,\infty)$. By Lemma C, Then $\bar{y}$ is a global attractor of all positive solutions of Eq.(1.1).

Case(2) $(p-q)y\bar{y}+qr(z-\bar{y})+r(y-z)+(q-p)z^{2}<0.$

So the function

$$\frac{pz+r+y}{qz+y}$$

is decreasing and

$$\max_{y\leq z\leq\bar{y}}\frac{r+y+pz}{qz+y}\frac{r+z+p\bar{y}}{q\bar{y}+z}=\frac{r+y+py}{qy+y}\frac{r+y+p\bar{y}}{q\bar{y}+\bar{y}}.$$

Also

$$\min_{\bar{y}\leq z\leq y}\frac{r+y+pz}{qz+y}\frac{r+z+p\bar{y}}{q\bar{y}+z}=\frac{r+y+py}{qy+y}\frac{r+y+p\bar{y}}{q\bar{y}+\bar{y}}.$$

Hence the function F is given by

$$F(y)=\frac{q}{\bar{y}}\frac{r+y+py}{qy+y}[\frac{r+y+p\bar{y}}{q\bar{y}+y}]^{k}.$$

Let

$$H(y)=\frac{q}{\bar{y}}\frac{r+y+py}{q\bar{y}+\bar{y}}[\frac{r+y+p\bar{y}}{q\bar{y}+y}]^{k}.$$

where $N=n+k$, n is a sufficiently large positive integer. From this we know that the function $H(y)\in C[(0,\infty),(0,\infty)]$ is strictly decreasing in $(0,\infty)$, and $H(\bar{y})=f(\bar{y})=\bar{y}$.

By a similar way to above, we can obtain that in this case $\bar{y}$ is a global attractor of all positive solutions of Eq.(1.1).

We then study the case ii)1<p < q. From Theorem 2.1 we see that every solution $\{y_n\}_{n=-1}^{\infty}$ of Eq.(1.1) is bounded and persists when p<q. Hence

$$\lambda = \liminf_{n\to\infty} y_n \quad \text{and} \quad \mu = \limsup_{n\to\infty} y_n ,$$

exist and are finite. So, for any positive number ε, there exists a positive integer N_0 such that

$$\lambda - \varepsilon < y_n < \mu + \varepsilon \quad \text{for} \quad n \ge N_0.$$

And moreover, at this time, the function $\dfrac{r+py+z}{qy+z}$ is

decreasing in y for $y, z \in (0,\infty)^2$. According to Eq.(1.1), one has, for $n \ge N_0 + 1$,

$$\frac{r+p(\mu+\varepsilon)+\lambda-\varepsilon}{q(\mu+\varepsilon)+\mu+\varepsilon} < y_{n+1} = \frac{r+py_n+y_{n-k}}{qy_n+y_{n-k}} < \frac{r+p(\lambda-\varepsilon)+\mu+\varepsilon}{(\lambda-\varepsilon)+\lambda-\varepsilon}.$$

There out, one can derive

$$\frac{r+p(\mu+\varepsilon)+\lambda-\varepsilon}{q(\mu+\varepsilon)+\mu+\varepsilon} \le \lambda \le \mu \le \frac{r+p(\lambda-\varepsilon)+\mu+\varepsilon}{(\lambda-\varepsilon)+\lambda-\varepsilon}.$$

In view of the arbitrary nature of ε, one has

$$\frac{r+p\mu+\lambda}{q\mu+\mu} \le \lambda \le \mu \le \frac{r+p\lambda+\mu}{q\lambda+\lambda}.$$

This indicates

$$r+p\mu+\lambda \le (q+1)\lambda\mu \le r+p\lambda+\mu.$$

So, $(p-1)\mu \le (p-1)\lambda.$. Noticing 1<p, we have $\mu \le \lambda.$. Again, $\mu \ge \lambda.$. Therefore, $\lambda = \bar{y} = \mu.$. The proof is over. For the global attractivity of the positive equilibrium point of Eq.(1.1), we also have the following results.

Theorem 3.3 Assume that $p, q, r \in (0,\infty)$. Then the unique positive equilibrium $\bar{y}$ of Eq.(1.1) is a global attractor provided that one of the following conditions is satisfied:

i) p < q and $\dfrac{r}{q-p} \le \dfrac{p}{q}$;

ii) p < q and $\dfrac{r}{q-p} \ge 1 + \dfrac{qr}{p(q+1)}$;

Proof. We first consider the case i) p<q and $\dfrac{r}{q-p} \le \dfrac{p}{q}$;. From Theorem 2.1, we

know $I=(\frac{p}{q},1+\frac{qr}{p(q+1)})$ is an invariant interval. At this time, we set

$$f_1(u,v)=\frac{r+pu+v}{qu+v}.$$

One can see the function f_1 is nonincreasing in x and nondecreasing in y. Eq.(1.1)can be written into

$$y_{n+1}=f_1(y_n,y_{n-k})=\frac{r+py_n+y_{n-k}}{qy_n+y_{n-k}},n=0,1,\cdots.$$

Then the system

$$x=f_1(y,x)=\frac{r+py+x}{py+x}$$

and

$$y=f_1(x,y)=\frac{r+px+y}{px+y}$$

has only one solution y=x.

So, by Lemma F, we see that the unique positive equilibrium of Eq.(1.1), then $\bar{y}$ is a global attractor.

We now consider Case ii). p < q and $\frac{r}{q-p}\geq 1+\frac{qr}{p(q+1)}$. We know $I=(\frac{p}{q},1+\frac{qr}{p(q+1)})$ is an invariant interval. Set

$$f_2(u,v)=\frac{r+pu+v}{qu+v}.$$

One can see the function f_2 is decreasing in both variables. From $f_2(m,m)=M$. and $f_2(M,M)=m.$, i.e.,

$$\frac{r+pm+m}{qm+m}=M$$

and

$$\frac{r+pM+M}{qM+M}=m,$$

we have m=M. So, all of the conditions in Lemma E are satisfied. Accordingly, in view of Lemma E, Eq.(1.1) has a unique positive equilibrium $\bar{y}$, and every solution of Eq.(1.1) converges to $\bar{y}$.The proof is complete for case ii).

References

[1] V. L. Kocic, G.Ladas, Global behavior of nonlinear difference equations of higher order with applications, Kluwer Academic Publishers, Dordrecht, 1993.

[2] M. R. S. Kulenovic, G. Ladas, Dynamics of Second Order Rational Difference Equations, with Open Problems and Conjectures, Chapman and Hall/CRC, 2002.

[3] Guangfeng Liu, Xianyi Li, Global attractivity of a nonlinear difference equation, Indian Journal of Mathematics, 50(3), (2008), 617-627.

[4] Youhui Su, Wantongh Li, Global attractivity of a higher order nonlinear difference equation, J. Difference Equ. Appl. 11(1), (2005), 947-958.

[5] Xianyi Li, Deming Zhu, Global asymptotic stability in a rational equation, J. Difference Equal. Appl., 9 (9), (2003), 833-839.

[6] Xianyi Li, Deming Zhu, Global asymptotic stability of a nonlinear recursive sequence, Appl. Math. Letters, 17(7), (2004), 833-838.

[7] Xianyi Li, Global behavior for a fourth-order nonlinear recursive sequence, J. Math. Anal. Appl., 2005, 312(2), 555-563