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SOME CHARACTERIZATIONS OF SEMI-OPEN, SEMI-CLOSED AND SEMI-CONTINUOUS MAPPINGS

Navpreet Singh Noorie and Jyoti Ahuja
Deptt. of Mathematics, Punjabi University, Patiala • E-mail : navresmath@yahoo.com

ABSTRACT
We obtain new characterizations of semi-open maps, semi-closed maps and semi-continuous maps using sets induced by the fibers. Semi-Continuity of (onto) maps is also characterized in terms of saturated sets.
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1. INTRODUCTION

It is well known that open mappings are characterized in terms of the interior operator and the closed mappings are characterized in terms of closure operator (see [4]). Interestingly in metric spaces open mappings were also characterized in terms of convergent sequences by Schochetman in [9]. Further open mappings as well as closed mappings were characterized both in terms of interior as well as in terms of the closure operator in [6], by using the sets of the form $f^{\#}(E)$. In particular, it is proved in [6] that

(i) a mapping is open if and only if $cl(f^{\#}(A)) \subset f^{\#}(cl(A))$.

(ii) a mapping is continuous if and only if $int(f^{\#}(A)) \subset f^{\#}(int(A))$.

It is therefore trivial that open maps also satisfy the condition $scl(f^{\#}(A)) \subset f^{\#}(cl(A))$ because $scl(A) \subset cl(A)$, where scl denotes the semi closure and continuous maps also satisfy the condition $int(f^{\#}(A)) \subset f^{\#}[cl(int(A))]$.We will see in Section 2 below that the relations $scl(f^{\#}(A)) \subset f^{\#}(cl(A))$ and $int(f^{\#}(A)) \subset f^{\#}[cl(int(A))]$ are in fact characterize semi-open maps and semi-continuous maps respectively which were studied by Biswas in 1969 [1] after Levine had introduced the semi-open sets and semi-continuity in [5].

In this paper, among other results we obtain new characterizations of semi-open maps in terms of semi-closures [Theorem 2.1 below], semi-closed maps in terms of semi-interiors [Theorem 2.5 below] and semi-continuous in terms of semi-interiors [Theorem 2.9 below].

Further semi-open (semi-closed) onto maps $f: X \to Y$ are described in terms of images under f of certain saturated sets in X [Corollary 2.3, 2.4] (Corollary 2.7, 2.8). Semi-continuity of onto maps is also characterized in terms of saturated sets [Theorem 2.11].

Subset A of a space X is semi-open [5] if A is contained in the closure of an open subset of A. The complement of semi-open set was defined as a semi-closed set in [3] and semi-closure and semi-interior were defined in a manner analogous to closure and interior, so that semi-interior of a set A is the union of all semi open sets contained in the set A and is therefore the largest semi-open set contained in A. Intersection of all semi-closed sets containing A is defined as a semi-closure and therefore it is the smallest semi-closed set containing A. Recently continuities on ideal minimal spaces and almost GPR continuity were separately investigated by S. Modak [2013] & S. Balasubramanian [2012]. Further semi-continuity was first studied by Levine [5] in 1963. A map $f: X \to Y$ is (i) semi-continuous [5] if inverse image of every open subset of Y is semi-open in X, (ii) semi-open [2] if image of every open subset of X is semi-open in Y, (iii) semi-closed [7] if image of every closed subset of X is semi-closed in Y.

Notation: For any sets X and Y, let $f: X \to Y$ be any map and E is any subset of X. Then

(i) $f^{\#}(E) = \{ y \in Y : f^{-1}y \subset E\}$, where $f^{-1}y$ denotes the fiber of y under f.

(ii) $E^{\#} = f^{-1}(f^{\#}(E))$

Throughout this paper X and Y will denote topological spaces on which no separation axioms are assumed. For a subset A of a space X, cl(A), int(A), sint(A), scl(A) and A^C will denote the closure of A, interior of A, semi-interior of A, semi-closure and complement of A in X respectively.
We will also make use of the following results:

Lemma 1.1 [6]: For any sets X and Y, let f: X→Y be any map and E be any subset of X. Then
(a) $E^\# = \{f^{-1}y: y \in Y \text{ and } f^{-1}y \subset E\} \subset E$;
(b) $f^\#(E^C) = [f(E)]^C$ and so $f^\#(E) = [f(E^C)]^C$ and $[f^\#(E^C)]^C = f(E)$;
(c) E is saturated if and only if $E = E^\#$;
(d) f is onto if and only if $f^\#(A) = f(A^\#)$ for each subset A of X;

Theorem 1.2 [5; Theorem1]: A subset A in a topological space X is semi-open if and only if $A \subset cl(int(A))$.
Since semi-closed set is the complement of semi-open set, so we have the following result.
Theorem 1.3. A subset A in a topological space X is semi-closed if and only if $int(cl(A)) \subset A$.

Theorem 1.4. For a subset A of a topological space X;
(i) [1] $sint(A) = A \cap [cl(int(A))]$
(ii) [8] $scl(B) = B \cup [int(cl(B))]$

Remark: From above theorem it is clear that $(scl(A))^C = sint(A^C)$.

2. RESULTS

As we discussed in the introduction open maps satisfy $scl(f^\#(A)) \subset f^\#(cl(A))$. But one can easily find many examples of the maps which satisfy $scl(f^\#(A)) \subset f^\#(cl(A))$ and are not open. However we see in the theorem below that the relation $scl(f^\#(A)) \subset f^\#(cl(A))$ in fact characterizes the semi-open maps.
Theorem 2.1: For any map f: X→Y, the following conditions are equivalent.
(a) f is a semi open map;
(b) For each subset A of X, $scl(f^\#(A)) \subset f^\#(cl(A))$;
(c) For each subset A of X, $int[cl(f^\#(A))] \subset f^\#(cl(A))$;
(d) For each closed subset F of X, $f^\#(F)$ is semi-closed in Y;
Proof: (a)⇒(b): Let y be a point of $scl(f^\#(A))$. By assumption, if V is an open set intersecting with the fiber of y, then f(V) is a semi-open set containing y and therefore must intersects with $f^\#(A)$. Let p be point in the intersection of f(V) and $f^\#(A)$. Let u be the point of V such that f(u)= p. But u must also be in A since the fiber of p is contained in A. Hence V intersects with A, which proves that y is in $f^\#(cl(A))$.
(b)⇒(c) follows from the Theorem 1.4 (ii)
(c)⇒(d): Let F be any closed subset of X. Then by (c) $int[cl(f^\#(F))] \subset f^\#(F)$. Hence $f^\#(F)$ is semi-closed by Theorem 1.3 and so (d) holds.
(d)⇒(a): Let U be any open subset of X. Then (c) implies that $f^\#(U^C)$ is semi-closed in Y. But $f^\#(U^C) = [f(U)]^C$,by Lemma 1.1(b). Therefore f(U) is semi open in Y. Hence f is a semi open map. This proves (a)

As an application of our first theorem we obtain the following characterizations of semi-open maps.
Corollary2.2: For a map f: X→Y. The following are equivalent:
(i) f is a semi-open map;
(ii) $f(int(A)) \subset cl[int(f(A))]$ for each subset A of X ;
(iii) $f(int(A)) \subset sint(f(A))$ for each subset A of X;
Proof: The proof follows from the equivalence of conditions (a), (c), (b) of the above theorem and the fact that $f^\#(E) = [f(E^C)]^C$ (Lemma 1.1(b)) and that $(scl(A))^C = sint(A^C)$ by the Remark.

The equivalence of (i) and (ii) in the above corollary is Theorem 9 of [2]

Since for onto maps $f^{\#}(A) = f(A^{\#})$ for each subset A of X; so we obtain the following characterisation of semi-open map.

Corollary2.3: Let f: X→Y be onto map. Then f is a semi-open map if and only if for each closed subset F of X, $f(F^{\#})$ is semi-closed in Y.

In the next Corollary, we see that image of saturated closed subset of X is semi-closed under semi-open surjection.

Corollary2.4: Let f: X→Y be semi- open onto map. Then for each saturated closed subset F of X, f (F) is semi-closed in Y. In particular, for any set A if $A^{\#}$ is closed , then $f(A^{\#})$ is semi-closed in Y.

In [6] it is shown that the relation $f^{\#}(\text{int}(A)) \subset \text{int}[f^{\#}(A)]$ characterizes closed mappings. So closed maps also satisfy $f^{\#}(\text{int}(A)) \subset \text{sint}[f^{\#}(A)]$ because $\text{int}(A) \subset \text{sint}(A)$ for any set A. We see in the Theorem below that the relation $f^{\#}(\text{int}(A)) \subset \text{sint}[f^{\#}(A)]$ in fact characterizes the semi-closed maps.

Theorem 2.5: For any map f: X→Y, following conditions are equivalent.

(a) f is semi-closed;

(b) For each subset A of X, $f^{\#}(\text{int}(A) \subset \text{sint}\,[f^{\#}(A)]$;

(c) For each subset A of X, $f^{\#}(\text{int}(A)) \subset \text{cl}[\text{int}(f^{\#}(A))]$;

(d) For each open subset U of X, $f^{\#}(U)$ is semi-open in Y;

Proof: (a)⇒(b): For y in $f^{\#}(\text{int}(A))$, there is an open set U such that $f^{-1}y \subset U \subset A$. Then since f is semi-closed, the set $G = (f(U^{C}))^{C}$ containing y is semi-open set. Therefore $f^{-1}(G) \subset U \subset A$ which shows y is in $\text{sint}[f^{\#}(A)]$.

(b)⇒(c) follows by the Theorem 1.4(i).

(c)⇒(d): Let U be any open subset of X, then (b) implies that $f^{\#}(U) \subset \text{cl}[\text{int}(f^{\#}(U))]$. Therefore $f^{\#}(U)$is an semi-open set by Theorem 1.2 and so (c) holds.

(d)⇒(a): Let F be any closed subset of X. Then (d) implies $f^{\#}(F^{C})$ is semi- open in Y and by Lemma 1.1(b), $f^{\#}(F^{C}) = [f(F)]^{C}$. Therefore f(F) is semi-closed in Y, and hence f is a semi-closed map.

Analogous to Corollary 2.2, we obtain the equivalent characterizations of the semi-closed mapping.

Corollary2.6: For any map f: X→Y. The following are equivalent.

(i) f is a semi-closed map;

(ii) $\text{scl}[f(A)] \subset f(\text{cl}(A))$, for each subset A of X ;

(iii) $\text{cl}[\text{int}(f(A))] \subset f(\text{cl}(A))$, for each subset A;

Since for onto maps $f^{\#}(A) = f(A^{\#})$ for each subset A of X; so we obtain the following characterisation of semi-closed map.

Corollary2.7: Let f: X→Y be onto map. Then f is a semi-closed map if and only if for each open subset U of X, $f(U^{\#})$ is semi-open in Y.

The next corollary gives the characterisation of semi -closed onto maps in terms of the images of saturated open subset of X.

Corollary2.8: Let f: X→Y be semi -closed onto map. Then for each saturated open set U of X, f(U) is semi-open in Y. In particular, for any set A, if $A^{\#}$ is open, then $f(A^{\#})$ is semi-open in Y.

We have the characterization of semi-continuity in terms of the inverse images of subsets of Y, but in the next theorem we will see the characterization of semi-continuity in terms of the images of subsets of X.

Theorem2.9: For any map f: X→Y, the following conditions are equivalent:

(a) f is semi-continuous;

(b) For each subset A of X, $\text{int}(f^{\#}(A)) \subset f^{\#}(\text{sint}(A))$;

(c) For each subset A of X, $\text{int}(f^{\#}(A)) \subset f^{\#}[\text{cl}(\text{int}(A))]$;

Proof: (a)⇒(b): For y in $\text{int}(f^{\#}(A))$, there exits an open set U such that $f^{-1}y \subset f^{-1}(U) \subset A$. Since f is semi continuous so we have $f^{-1}y \subset f^{-1}(U) \subset \text{cl}[\text{int}(f^{-1}(U))] \subset \text{cl}(\text{int}(A))$. This implies $f^{-1}y \subset A \cap \text{cl}(\text{int}(A)) = \text{sint}A$ by Theorem 1.4(i); which means y is in $f^{\#}(\text{sint}A)$.

(b)⇒(c) follows by Theorem 1.4(ii)

(c)⇒(a): By assumption for any subset A of X and for any open set H in Y; whenever $f^{-1}(H) \subset A$, then $f^{-1}y \subset cl(int(A))$ for all $y \in H$. Let G be an open set then taking $f^{-1}(G) = A$ we have for every y in G, $f^{-1}y \subset cl[int(f^{-1}(G))]$. Therefore $f^{-1}(G) \subset cl[int(f^{-1}(G))]$ which proves the semi-contiunuity of f.

As in the case of semi-open maps before, we obtain the following characterizations of the semi-continuous maps by using Lemma 1.1(b).

Corollary2.10: For any map f: X→Y, the following conditions are equivalent:

(i) f is a semi-continuous map;

(ii) $f[cl(int(A)] \subset cl(f(A))$ for each subset A of X;

(iii) $f(sint(A)) \subset cl(f(A))$ for each subset A of X.

The following theorem characterizes semi-continuity for onto maps.

Theorem2.11: Let f: X→Y be any onto map. Then the following conditions are equivalent.

(a) f is semi-continuous;

(b) $int[f(A^{\#})] \subset f[(sint(A))^{\#}]$ for any subset A of X;

(c) $int[f(A^{\#})] \subset f[(cl(int(A)))^{\#}]$ for any subset A of X;

(d) $A^{\#}$ is semi-open in X whenever $f(A^{\#})$ is open in Y;

(e) For any saturated set E in X, E is semi-open in X whenever f (E) is open in Y;

(f) For any saturated set E in X, E is semi-closed in X whenever f (E) is closed in Y.

Proof: The equivalence of (a),(b) and (c) follows from the above theorem and Lemma 1.1(d).

(a)⇒(d) follows from Lemma 1.1(d) and the fact that if f: X→Y be semi-continuous and E be any subset of X. Then $E^{\#}$ is semi-open in X whenever $f^{\#}(E)$ is open in Y.

(d)⇒(e): Since E is saturated if and only if $E = E^{\#}$, therefore (e) follows from (d) for arbitrary f.

(e)⇒(f): Since f is onto and E is saturated we have $[f(E)]^C = f(E^C)$ so f(E) closed implies that $f(E^C)$ is open and therefore E^C is semi-open in X by (e) Hence E is semi-closed in X, proving that (f) holds.

(f)⇒(a): For any closed Subset S of Y. $S = f(f^{-1}(S))$ implies by (f) that $f^{-1}(S)$ is semi-closed in X. Hence f is semi-continuous.

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