ON UNIFIED CLASS OF α - SPIRALLIKE FUNCTIONS OFCOMPLEX ORDER

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*ABSTRACT.*In this paper, we consider an unified class of α - spirallike functions of complex order. Necessary and sufficient condition for functions to be in this class is obtained.Some of our results generalize previously known result.

1.INTRODUCTION

Let A denote the class of all analytic function of the form

(1.1) $$f(z) = z + \sum_{n=1}^{\infty} a_n z^n$$

in the open unit disc $\mathrm{E} = \{ z \in \mathrm{C} : |z| < 1 \}$. Let S be the subclass of A consisting of univalent functions. Also, we denote by S^*, C and K the familiar subclasses of A consisting of functions which are respectively starlike, convex and close-to-convex in E. Our favorite references of the field are [4, 5] which covers most of the topics in a lucid and economical style.

For $-\pi/2 < \alpha < \pi/2$, a function $f \in A$ is said to be α-spiral in E if

(1.2) $$Re\left\{ e^{i\alpha} \frac{z f'(z)}{f(z)} \right\} > 0, \qquad (z \in \mathrm{E}).$$

Similarly, a function $f \in \mathrm{A}$ is said to be convex α-spirallike in E if

(1.3) $$Re\left\{ e^{i\alpha} \left(1 + \frac{z f''(z)}{f'(z)} \right) \right\} > 0, \qquad (z \in \mathrm{E}).$$

We denote α-spirallike functions and convex α-spirallike functions respectively by $SP(\alpha)$ and $CSP(\alpha)$. If f is in $CSP(\alpha)$, then it does not follow that $f(\mathrm{E})$ is convex or even spirallike in shape. Also, we note that functions in $CSP(\alpha)$ need not be univalent whereas functions in $SP(\alpha)$ are univalent.

Suppose if f and g are analytic in E, we say that f is subordinate to g written symbolically as $f \prec g$, if there exist a schwarz function w in E such that $f(z) = g(w(z)), z \in \mathrm{E}$. If g is univalent in E, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathrm{E}) \subset g(\mathrm{E})$. That is $f \prec g$ will mean that every value taken by f in E is also taken by g.

Let $\phi(z)$ be an analytic function with positive real part on ϕ with $\phi(0) = 1, \phi'(0) > 0$ which maps the unit disc E onto a region starlike with respect to 1 which is symmetric with respect to the real axis.

Let $\mathrm{S}^*(\phi)$ be the class of functions in $f \in \mathrm{S}$ for which

(1.4) $$\frac{z f'(z)}{f(z)} \prec \phi(z).$$

and $\mathrm{C}(\phi)$ class of functions in $f \in \mathrm{S}$ for which

(1.5) $$1 + \frac{z f'(z)}{f(z)} \prec \phi(z).$$

The classes $\mathrm{S}^*(\phi)$ and $\mathrm{C}(\phi)$ were introduced and studied by Ma and Minda [7]. Analogous to the classes $\mathrm{S}^*(\phi)$ and $\mathrm{C}(\phi)$, Ravichandran et. al.[10] considered the classes $\mathrm{S}_b^*(\phi)$ and $\mathrm{C}_b(\phi)$ of complex order $b(b \in \mathrm{C} \setminus \{0\})$ which is defined as follows:

(1.6) $$\mathrm{S}_b^*(\phi) = \left\{ f \in \mathrm{A} : 1 + \frac{1}{b}\left(\frac{z f'(z)}{f(z)} - 1 \right) \prec \phi(z) \right\},$$

and

(1.7) $$\mathrm{C}_b(\phi) = \left\{ f \in \mathrm{A} : 1 + \frac{1}{b} \frac{z f''(z)}{f'(z)} \prec \phi(z) \right\}.$$

From (1.6) and (1.7) we have

$$f \in C(\phi; b) \Leftrightarrow z f' \in S^*(\phi; b).$$

Now, we introduce a more general class of α-spirallike function of complex order $\tau^\alpha(\phi; m, b)$ as follows.

Definition 1. *The class $\tau^\alpha(\phi; m, b)$ of functions $f \in A$ analytic in E given by (1.1) and satisfying the condition*

(1.8) $$1+\frac{e^{i\alpha}}{b\cos\alpha}\left(\frac{z f^{(m+1)}(z)}{f^{(m)}(z)}-1+m\right) \prec \phi(z), \quad (z \in E)$$

where $-\pi/2 < \alpha < \pi/2, b \in C \setminus \{0\}$ and m is a positive integer.

We note that by specializing $b, m, \alpha, \phi(z)$ in the function class $\tau^\alpha(\phi; m, b)$, we obtain several well-known and new subclasses of analytic functions. Here we list a few of them:

(i) $\tau^\alpha(\frac{1+z}{1-z}; 0, b) = S^\alpha(b)$ and $\tau^\alpha(\frac{1+z}{1-z}; 1, b) = C^\alpha(b)$, (Al-Oboudi and Haidan [1] and Aouf et.al. [2]).

(ii) $\tau^0(\phi; 0, b) = S^*(\phi; b)$ and $\tau^0(\phi; 1, b) = C(\phi; b)$, (Ravichandran et. al. [10]).

(iii) $\tau^0(\phi; 0, 1) = S^*(\phi)$ and $\tau^0(\phi; 1, 1) = C(\phi)$ (Ma and Minda [7]).

2.MAIN RESULTS

To prove our main result, we cite the following lemma.

Lemma 1.*([12])Let ϕ be a convex function defined on $E, \phi(0) = 1$. Define F(z) by*

(2.1) $$F(z) = z\exp\left(\int_0^z \frac{\phi(t)-1}{t}\,dt\right).$$

Let p(z) = 1 + $p_1 z + p_2 z^2$+..... be analytic in E, then

(2.2) $$1+\frac{z p'(z)}{p(z)} \prec \phi(z)$$

if and only if for all $|s|\le 1$ and $|t|\le 1$ we have

(2.3)
$$\frac{p(tz)}{p(sz)} \prec \frac{s\,F(tz)}{t\,F(sz)}$$

Theorem 1. *Let $F(z)$ be defined as in (9) and let $\phi(z)$ be a convex function in* E *with $\phi(0)=1$. The function $f\in\tau^{\alpha}(\phi; m, b)$ if and only if for all $|s|\le 1$and $|t|\le 1$ we have*

(2.4)
$$\left[\left(\frac{t}{s}\right)^{m-1}\frac{f^{(m)}(tz)}{f^{(m)}(sz)}\right]^{\frac{e^{i\alpha}}{b\cos\alpha}} \prec \frac{s\,F(tz)}{t\,F(sz)}$$

Proof. Let $p(z)$ be defined by

(2.5)
$$p(z)=\left[\frac{f^{(m)}(z)}{z^{1-m}}\right]^{\frac{e^{i\alpha}}{b\cos\alpha}} \quad (z\in \mathrm{E})$$

Taking logarithmic derivative of (2.5), we get

$$1+\frac{zp'(z)}{p(z)}=1+\frac{e^{i\alpha}}{b\cos\alpha}\left(\frac{zf^{(m+1)}(z)}{f^{(m)}(z)}-1+m\right).$$

Since $f\in\tau^{\alpha}(\phi; p, m, b)$ we have

$$1+\frac{zp'(z)}{p(z)}\prec\phi(z)$$

and the result now follows from lemma1.

Corollary 1. *Let $F(z)$ be defined by*

$$F(z)=z\exp\left[\int_0^z \frac{1}{t}\left\{\frac{\gamma-\beta}{\pi} \quad i\log\left(\frac{1-e^{2\pi i(1-\beta)\backslash(\gamma-\beta)}w(t)}{1-w(t)}\right)\right\}dt\right].$$

The function $f\in$ A satisfies the condition

$$\beta < Re\left\{1+\frac{e^{i\alpha}}{b\cos\alpha}\left(\frac{zf^{(m+1)}(z)}{f^{(m)}(z)}-1+m\right)\right\} < \gamma$$

if and only if for all $|s|\leq 1$and $|t|\leq 1$ we have

$$\left[\left(\frac{t}{s}\right)^{m-1}\frac{f^{(m)}(tz)}{f^{(m)}(sz)}\right]^{\frac{e^{i\alpha}}{b\cos\alpha}} \prec \frac{s\,F(tz)}{t\,F(sz)}$$

Proof. In Definition 1, let $\phi(z)$ be defined by

$$\phi(z)=1+\frac{\beta-\alpha}{\pi}\quad i\log\left(\frac{1-e^{2\pi i(1-\alpha)\backslash(\beta-\alpha)}w(z)}{1-w(z)}\right).$$

Clearly $\phi(z)$ is analytic which maps E onto a convex domain conformally with $\phi(0)=1$. Using (1.8) together with Theorem 1, proves the result.

Corollary 2. *Let* $F(z)$ *be defined by*

$$F(z)=z\exp\left(\int_0^z\frac{\phi(t)-1}{t}dt\right).$$

The function $f\in$ A satisfies the condition

$$1+\frac{e^{i\alpha}}{b\cos\alpha}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\phi(z)$$

if and only if for all $|s|\leq 1$and $|t|\leq 1$ we have

$$\left[\frac{s\,f(tz)}{t\,f(sz)}\right]^{\frac{e^{i\alpha}}{b\cos\alpha}} \prec \frac{s\,F(tz)}{t\,F(sz)}.$$

Corollary 3. *Let* $F(z)$ *be defined by*

$$F(z)=z\exp\left(\int_0^z\frac{\phi(t)-1}{t}dt\right).$$

The function $f \in \mathrm{A}$ satisfies the condition

$$1+\frac{e^{i\alpha}}{b\cos\alpha}\left(\frac{z f^{''}(z)}{f^{'}(z)}\right) \prec \phi(z)$$

if and only if for all $|s| \leq 1$and $|t| \leq 1$ we have

$$\left[\frac{f^{'}(tz)}{f^{'}(sz)}\right]^{\frac{e^{i\alpha}}{b\cos\alpha}} \prec \frac{s F(tz)}{t F(sz)}$$

Remark 2.1If $\alpha = 0$, then the Corollary 2 and Corollary 3 reduces to well-known result proved by Shanmugam et al. in [11].

Lemma 2.*([13]) Let q(z) be a univalent in* E *and let* $\varphi(z)$ *be analytic in a domaincontaining* $q(\mathrm{E})$*. If* $\frac{z q^{'}(z)}{q(z)}$ *is starlike, then*

$$z p^{'}(z)\varphi(p(z)) \prec z q^{'}(z)\varphi(q(z)),$$

then p(z) $\prec$ q(z)and q(z) is best dominant.

Theorem 2. *Let* $\phi(z)$ *be a starlike with respect to 1 and F(z)given by (2.1) be starlike.If* $f \in \tau^{\alpha}(\phi;\ p,\ m, b)$ *then we have*

(2.6)
$$\left(\frac{z^m f^m(z)}{z^p}\right) \prec \left(\frac{F(z)}{z}\right)^{\frac{b\cos\alpha}{e^{i\alpha}}}$$

Proof. Let $p(z)$ be given by (2.5) and $q(z)$ be given by

(2.7)
$$q(z) = \frac{F(z)}{z} \quad (z \in \mathrm{E}).$$

After a simple computation we obtain

$$1+\frac{z\,p'(z)}{p(z)}=1+\frac{e^{i\alpha}}{b\cos\alpha}\left(\frac{z\,f^{(m+1)}(z)}{f^{(m)}(z)}-p+m\right).$$

and

$$\frac{z\,q'(z)}{q(z)}=\frac{z\,F'(z)}{F(z)}-1=\phi(z)-1.$$

Since $f\in\tau^{\alpha}(\phi;\,p,\,m,\,b)$, we have

$$\frac{z\,p'(z)}{p(z)}\prec\frac{z\,q'(z)}{q(z)}.$$

and the result now follows from Lemma 2

Corollary 4. *Let* b *be a non zero complex number. If* $f\in \mathrm{A}$*, and*

$$1+\frac{e^{i\alpha}}{b\cos\alpha}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+z}{1-z},$$

then

$$\frac{f(z)}{z}\prec(1-z)^{-2be^{-i\alpha}\cos\alpha}$$

and $(1-z)^{-2be^{-i\alpha}\cos\alpha}$, is the best dominant.

Corollary 5. *Let* b *be a non zero complex number. If* $f\in \mathrm{A}$*, and*

$$1+\frac{e^{i\alpha}}{b\cos\alpha}\frac{z\,f''(z)}{f'(z)}\prec\frac{1+z}{1-z},$$

then

$$f'(z)\prec(1-z)^{-2be^{-i\alpha}\cos\alpha}$$

and $(1-z)^{-2be^{-i\alpha}\cos\alpha}$, is the best dominant.

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