ON HORIZONTAL AND COMPLETE LIFTS OF (1, 1) TENSOR FIELD *f* SATISFYING STRUCTURES $f^{11} - {}^{2} f^{9} = 0$ AND $f^{10} - f = 0$

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Abstract. The horizontal and complete lifts from a differentiable manifold of class C^{∞} to its co-tangent bundle $T^{*}(M^{n})$ have been studied by Yano and Patterson [4, 5]. Yano and Ishihara [6] studied lifts of an *f*-structure in the tangent and co-tangent bundle. *f*-structures manifolds of degree 8 have been studied by Kim, J.B. [2]. The present paper deals with some problems on horizontal arid complete lifts of structures mentioned above in tangent and co-tangent bundles and the prolongation in the second tangent space $T_2(M^n)$. Integrability conditions of *f*-structure manifolds of degree 10 in tangent bundle have also been discussed.

1. Preliminaries

Let M^n be *n*-dimensional differentiable manifold of class C^{∞}. Let $T^{*}(M^{n})$ be the co-tangent bundle of M^n. Then $T^{*}(M^{n})$ is also a differentiable manifold of class C^{∞} and of dimension 2*n*. Throughout this chapter, we make use of the following notations and conventions:

(i) The map π: $T^{*}(M^{n})$ M^n is the projection map of $T^{*}(M^{n})$ onto M^n.

(ii) Suffixes *a*, *b*, *c*,....., *h*, *i*, *j* take the values of 1 to *n* and = *i* + *n*. Suffixes *A*, *B*, *C*,....... take the values 1 to 2*n*.

(iii) (M^n) is the set of tensor fields of class C^{∞} and type (*r*, *s*) on M^n. Similarly, $(T^{*}(M^{n}))$ denotes such tensor fields in $T^{*}(M^{n})$.

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(iv) Vector fields in M^n are denoted by *X*, *Y*, *Z*,.... and their Lie derivative by L_X. The Lie product of *X* and *Y* is denoted by [*X*, *Y*]. If *A* is a point in M^n, $\pi^{-1}(A)$ $T^{*}(M^{n})$ called fibre over *A*. Any point *P* $\pi^{-1}(A)$ can be denoted by ordered pair (A, P_A), P_A is the value of 1-form *p* at *A*. If *U* be a co-ordinate neighborhood in M^n with co-ordinates (X^h), $\pi^{-1}(U)$ is coordinate neighborhood on $T^{*}(M^{n})$ with co-ordinate functions (X^h, P_i). If *P* lies in the intersecting region $\pi^{-1}(U)$ $\pi^{-1}(U')$ with co-ordinate functions (X^h, P_i) and $(X^{h'}, P_{i},)$, then $X^{h'} = X^{h'}(X^h)$ and = P_i.

Then we have [5]

$$(X + Y)^C = X^C + Y^C \qquad (1.1)$$

and

$$(f)^C(Z)^C = (f\,Z)^C + (L_Z\, f)^V \qquad (1.2)$$

Let M^n be an *n*-dimensional connected differentiable manifold of class C^{∞}. Let there be given in M^n, a (1, 1) tensor field *f* of class C^{∞} satisfying

$$f^{11} - {}^{2\,n} f^{9} = 0, \qquad (1.3)$$

where is non-zero complex number.

Also,

$$\text{rank}\,(f) = (\text{rank}\, f^{9} + \dim M^{n}) = r(\text{a constant everywhere on } M^{n})$$

Let the operators l^* and m^* be defined as

$$l^* = \text{ and } m^* = I - , \qquad (1.4)$$

where I denotes the identity operator on M^n, Then the operators l^* and m^* applied to the tangent space at a point of the manifold are complementary projection operators. We call such a structure as f(11, 9)-stucture of rank r on M^n.

1.1. **Agreement.** In what follows we make use of the following results [6]. For any X, Y (M^n), we have

(i) $[X^C, Y^C] = [X, Y]$

(ii) $f^C X^C = [f X]$.

Definition 1.1. *Let f be a non-zero tensor field of type* (1, 1) *and of class* C^∞ *on an n-dimensional manifold* M^n *such that* [2]

$$f^{10} - f = 0, \qquad (1.5)$$

where rank of f is constant everywhere and equal to r.

Let the operators on M^n be defined as follows [2]

$$l = f^9 \text{ and } m = I - f, \qquad (1.6)$$

where I denotes the identity operator. From the operators denned by (1.6), we have

(1.7)

For f satisfying (1.5), there exist complementary distributions L and M corresponding to the projection operators l and m respectively. If rank(f) be r. constant on M^n then dim$L = r$ and dim$M = n - r$. We have the following results:

$$f\,l = l\,f = f \text{ and } f\,m = m\,f = 0, \qquad (1.8)$$

$$f\,l = 1 \text{ and } fm = 0. \qquad (1.9)$$

Let us call such a structures as f-structure of degree 10.

2. The complete of f in the tangent bundle $T(M^n)$

The complete lift of f^C of an element of (M^n) with local component of has components of the form

$$f^C = \qquad (2.1)$$

Now, we prove some theorems on the complete lifts of f(11-, 9)-structure satisfying (1.3) and also its integrability conditions.

Theorem **2.1.** *The complete lift of* (1,1) *tensor field f satisfying f*(11, 9)-*structure in* M^n *will admit the similar structure in the tangen bundle* $T(M)^n$.

Proof. Let f, g (M^n), then we have

$$(f\,g)^C = f^C g^C. \qquad (2.2)$$

Putting f = g, we obtain

$$(f^2)^C = (f^C)^2. \qquad (2.3)$$

Putting $g = f^2$ in (2.2) and making use of (2.3), we get

$$(f^3)^C = (f^C)^2. \qquad (2.4)$$

Continuing the above process of replacing g in equation (2.2) by higher degree of f, we obtain

$$(f^{10})^C = (f^C)^{10} \text{ and so on.}$$

Taking complete lift on both sides of equation (1.3), we get

$$(f^{11})^C - (^2f^9)^C = 0$$

which in view of the equation (2) gives

$$(f^C)^{11} - ^2(f^C)^9 = 0. \qquad (2.5)$$

Thus, the complete lift of f also has f(11, 9)-structure in $T(M^n)$. The complete lift $(l^*)^C$ and $(m^*)^C$ of l^* and m^* are complementary projection tensors in $T(M^n)$. Thus, there exist in $T(M^n)$ two complementary distribution $(L^*)^C$ and $(M^*)^C$ determined by $(l^*)^C$ and $(m^*)^C$ respectively.

Theorem 2.2. *The complete lift* $(m^*)^C$ *of the distribution* M^* *in* $T(M^n)$ *is integrable if and only if* M^* *is integrable in* M^n.

Proof. It is well known that the distribution M^* is the integrable in M^n if and only if

$$l^*[m^* X, m^* Y] = 0. \quad (2.6)$$

Taking complete lift of on both side of equation (2.6), we get

$$(l^*)^C[(m^*)^C X^C, (m^*)^C Y^C] = 0, \quad (2.7)$$

where

$$(l^*)^C = (I - m^*)^C = I - (m^*)^C, \text{ as } I^C = I.$$

In consequence of equation (2.7), $(m^*)^C$ is integrable in $T(M^n)$.

Theorem 2.3. *The complete lift* $(l^*)^C$ *of the distribution* L^* *in* $T(M^n)$ *is integrable if and only if* L^* *is integrable in* M^n.

Proof. Proof is same as that of the theorem **2.2.**

Theorem 2.4. *The structure* f^C *is partially integrable if and only if f is partially integrable in* M^n.

Proof. We know that *f* is partially integrable if and only if

$$N(l^* X, l^* Y) = 0. \quad (2.8)$$

Taking complete lift on both sides, we obtain

$$N((l^*)^C X^C, (l^*)^C Y^C) = 0. \quad (2.9)$$

Hence, f^C is partially integrable if and only if *f* is partially integrable in M^n.

Theorem 2.5. *For any X, Y* (M^n), *let f be integrable in* M^n. *Thus,* f^c *is integrable m* $T(M^n)$ *if and only if* $N^C(X^C, Y^C) = 0$.

Proof. We know that *f* is integrable if and only if

$$N(X, Y) = 0, \quad (2.10)$$

where $N(X, Y)$ *is* the Nijenhuis tensor of *f* satisfying (1.3) and it is given by [6]

$$N_{f,f}(X, Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y]. \quad (2.11)$$

Taking complete lift on both sides, we have

$$N^C(X^C, Y^C] = [f^C X^C, f^C Y^C] - f^C[f^C X^C, Y^C] - f^C[X^C, fY^C] + (f^2)^C[X^C, Y^C]. \quad (2.12)$$

Also, taking complete lift of (2.10), we get

$$N^C(X^C, Y^C) = 0,$$

which in view of equation (2,11) and (2.12) and the fact *f* is integrable in M^n shows that f^C is integrable in $T(M^n)$.

3. **The complete lift of** *f*(11, 9)**-structure in cotangent bundle**

In this section, we prove some theorems on complete lift of *f* satisfying *f*(11, 9)-structure.

Theorem 3.1. *The Nijenhuis tensor of the complete of* f^{11} *vanishes if the lie derivative of the tensor field* f^{11} *with respect to X and Y are both zero and f is an almost -structure on* M^n.

Proof. In consequence of (2.11), the Nijenhuis tensor of $/^n$ is given by

$$\begin{aligned}(X^C, Y^C) = {} & [(f^{11})^C X)^C, (f^{11})^C Y)^C] \\ & - (f^{11})^C[(f^{11})^C X^C, X^C] \\ & - (f^{11})^C [X^C, (f^{11})^C Y^C] \\ & + (f^{11})^C(f^{11})^C [X^C, X^C] \end{aligned} \quad (3.1)$$

which in view of (1.3) takes the form

$$(X^C, Y^C) = {}^4[(f^9)^C X^C, (f^9)^C Y)^C] - {}^4(f^9)^C[(f^9)^C X^C, X^C] - {}^4(f^9)^C [X^C, (f^9)^C Y)^C] + {}^4(f^9)^C (f^9)^C [X^C, X^C] \quad (3.2)$$

In consequence of (1.2), we have

$$(f^9)^C X^C = (f^9 X)^C + (L_X f^9)^V. \quad (3.3)$$

Hence, we get

$$(X^C, Y^C) = {}^4\{[f^9 X)^C, (f^9) Y)^C] + [(L_X f^9)^V, (f^9 Y)^V] + [(f^9 X)^C, (L_Y f^9)^V] + [(L_X f^9)^V, (L_Y f^9)^V] - (f^9)^C [(f^9 X)^C, Y^C] - (f^9)^C [(L_X f^9)^V, Y^C] - (f^9)^C [(X^C, (f^9 Y)^C - (f^9)^C [X^C, (L_Y f^9)^V] + (f^9)^C (f^9)^C [(X^C, Y^C]\}. \quad (3.4)$$

If the lie derivatives of the tensor field f^9 with respect to X and Y are both zero, we have

$$L_X f = 0 \text{ and } L_Y f = 0.$$

Therefore, equation (3.4) takes the form

$$(X^C, Y^C) = {}^4\{[f^9 X)^C, (f^9 Y)^C - (f^9)^C [(f^9 X)^C, Y^C] - (f^9)^C [X^C, (f^9 Y)^C] + (f^9)^C (f^9)^C [(X^C, Y^C]\}. \quad (3.5)$$

$$(X^C, Y^C) = {}^4\{[f^9 X, f^9 Y]^C - (f^9)^C [f^9 X, Y]^C - (f^9)^C [X^C, f^9 Y]^C + (f^9)^C (f^9)^C [X^C, Y]^C\}. \quad (3.6)$$

Let f be an almost -structure on M^n, then $f^2 = {}^2 I$, where I is the unit tensor field. Hence, f^9 = I and therefore (3.6) takes the form

$$(X^C, Y^C) = {}^4\{[X, Y]^C - [X, Y] - [X, Y]^C + [X, Y]^C\} = 0$$

Theorem 3.2. *The Nijenhuis tensor of the complete of f^{11} is equal to ^{4} multiplied by the complete lift of the Nijenhuis tensor of f^{11} if*

i. $L_X f^9 = 0$, $L_Y f^9 = 0$,

ii. $[X, Y]^C = 0, = 0,$

where $= f^9 + f^9 - f^{18}$.

Proof. In view of equation (1.1) and (2.11), we have

$$= (X, Y)^C = [f^9 X, f^9 Y]^C - (f^9[f^9 X, Y])^C - (f^9 [X, f^9 Y]^C + (f^{18}[X, Y])^C, \quad (3.7)$$

which on account of (3.3) yields

$$= (X, Y)^C = [f^9 X, f^9 Y]^C - (f^9)^C [f^9 X, Y]^C - [f^9]^V - (f^9)^C [X, f^9 Y]^C - [f^9]^V - (f^{18})^C [X, Y]^C - [f^{18}]^V.$$

But, we have [6]

$$(f^9)^C (f^9)^C = (f^{18})^C + ()^V. \qquad (3.8)$$

Hence in view (3.8), the equation (3.7) becomes

$$(X, Y)^C = [f^9 X, f^9 Y]^C - (f^9)^C [f^9 X, Y]^C - (f^9)^C [X, f^9 Y]^C - (f^{18})^C [X, Y]^C - [f^9]^V - [f^9]^V - [f^{18}]^V. \qquad (3.9)$$

Now, from (3.8), we have

$$(f^{18})^C = (f^9)^C (f^9)^C - ()^V.$$

Thus,

$$(X, Y)^C = [f^9 X, f^9 Y]^C - (f^9)^C [f^9 X, Y]^C - (f^9)^C [X, f^9 Y]^C - (f^9)^C (f^9)^C [X, Y]^C - ()^V [X, Y]^C - [f^9]^V - [f^9]^V - [f^{18}]^V. \qquad (3.10)$$

In view of the equation (3.10), the equation (3.5) takes the form

$$(X^C, Y)^C = {}^4 \{(X, Y)^C + ()^V [X, Y]^C - \{f^9 + f^9 + [f^{18}\}^V\}.$$

In consequence of ( ), we have

$$(X^C, Y)^C = {}^4 \{(X, Y)^C + ()^V [X, Y]^C\} - 1 . \qquad (3.11)$$

Let $[X, Y]^C = 0$ and $= 0$, the (3.11) reduce to

$$(X^C, Y)^C = {}^4 ((X, Y)^C).$$

Theorem 3.3. *The Nijenhuis tensor of the complete of* f^{11} *is equal to the complete lift of the Nijenhuis tensor of* f^{11} *if*

i . $L_X f^9 = 0$, $L_Y f^9 = 0$,

ii. $L_X Y = 0$, $= 0$.

Proof. Since $[X, Y]^C = 0$ implies that $[X, Y] = 0$ or $L_X Y = 0$. Therefore from (3.2), the results follows.

Theorem 3.4. *The process of computing the Nijenhuis tensor of* f^9 *and taking complete lift are commutative.*

Proof. Theorem follows easily from the equation (3.1) and theorem 3.3.

4. The horizontal lift of a $f(11,9)$-structure

In this section, we prove theorem on horizontal lift satisfying the structure (1.3).

Theorem 4.1. *Let* f $(M^n))$ *be a* $f(11,9)$ *-structure in* M^n, *then the horizontal lift* f^H *off is also* $f(9, 7)$*-structure on* $T^*(M^n)$.

Proof. For every f, g (M^n), we have [6]

$$f^H g^H + g^H f^H = (fg + g f)^H \qquad (4.1)$$

Putting $g = f$, we get

$$2(f^H)^2 = (2 f^2)^H$$

or

$$(f^H)^2 = (f^2)^H \qquad (4.2)$$

Replacing g by f^2 in (4.1), we get

$$(f^H) (f^2)^H + (f^2) (f^H) = (2f^3)^H$$

which in view of (4.2) yields

$$(f^H)^3 + (f^H)^3 = (2f^3)^H$$

i.e.,

$$(f^H)^3 = (f^3)^H.$$

Continuing this process and replacing g by $f^4, f^5, f^6, f^7, f^8, f^9$, we get

$$(f^H)^{10} = (f^{10})^H.$$

Also,

$$(f^H)^9 = (f^9)^H \qquad (4.3)$$

And

$$(f^H)^{11} = (f^{11})^H \qquad (4.4)$$

Since f is a $f(11, 9)$-structure on M^n, therefore

$$f^{11} - {}^2 f^9 = 0.$$

Hence, from (4.3) and (4.4), we get

$$(f^H)^{11} = (f^{11})^H = {}^2 (f^9)^H = {}^2 (f^H)^9$$

Or

$$(f^H)^{11} - {}^2 (f^H)^9 = 0.$$

Thus, f^H is a f (11,9)-structure on $T^*(M^n)$.

5. **Prolongation of a** $f(11, 9)$-structure **in second tangent space** $T_2(M^n)$

Let us denote $T_2(M^n)$, the second order tangent bundle over M^n and let f^H be the second lift on f in $T_2(M^n)$. Then, we have for any f, g (M^n)), the following holds

$$\begin{aligned} (g^{II} f^{II}) X^{II} &= g^{II}(f^{II} X^{II}) \\ &= g^{II}(f X)^{II} \\ &= (g (fX))^{II} \\ &= (g f)^{II} X^{II} \qquad (5.1) \end{aligned}$$

for every X (M^n), therefore we have

$$g^{II} f^{II} = (g f)^{II} \quad = g^{II}(f^{II} X^{II})$$

If $P(t)$ denotes a polynomial of variable t, then we have

$$(P(f))^{II} = P(f)^{II}, \qquad (5.2)$$

where f (M^n),.

Theorem 5.1. *The second lift* f^{II} *defines a* $f(11,9)$-*structure in* $T_2(M^n)$, *if and only if f defines a f* (9,7)-*structure in* M^n.

Proof. Let f satisfy (1.3), then f defines a $f(11,9)$-structure in M^n satisfying

$$f^{11} - {}^2 f^9 = 0,$$

which in view of equation (5.2) takes the form

$$(f^{II}) f^{11} - {}^2 (f^{II})^9 = 0. \qquad (5.3)$$

Therefore, f^{II} defines a $f(11,9)$-structure on $T_2(M^n)$.

Theorem 5.2. *The second lift* f^{II} *is integrabte in* $T_2(M^n)$ *if and only if f is integrable in* M^n.

Proof. Let us denote N^{II} and N, the Nijenhuis tensors of f^{II} and f respectively. Then we have [6]

$$N^{II}(X, Y) = (N(X, Y))^{II}. \qquad (5.4)$$

We know that $f(11, 9)$-structure is integrable in M^n if and only if

$$N(X, Y) = 0$$

which in view of (5.4) is equivalent to

$$N^{II}(X, Y) = 0. \qquad (5.5)$$

Thus, N^{II} *is* integrable if and only if f is integrable in M^n.

Theorem 5.3. *The second lift f^{II} off is partially integrable in $T_2(M^n)$ if and only if f is partially integrable in M^n.*

Proof. We know that for f to be partially integrable in M^n, the following holds

$$N(l^* X, l^* Y) = 0$$

and

$$N(m^* X, m^* Y) = 0.$$

which in view of equation (5.4) takes form

$$N^{II}((l^*)^{II} X^{II}, (l^*)^{II} Y^{II}) = 0 \qquad (5.6)$$

and

$$N^{II}((m^*)^{II} X^{II}, (m^*)^{II} Y^{II}) = 0, \qquad (5.7)$$

where $(l^*)^{II}$ and $(m^*)^{II}$ are operators in $T_2(M^n)$ which defines the distributions $(L^*)^{II}$ and $(M^*)^{II}$ respectively. Thus, the equations (5.6) and (5.7) gives the condition for f^{II} to be partially integrable. The converse of theorems **5.2** and **5.3** follows in the similar manner.

6. Integrability conditions of *f*-structure in a tangent bundle

Let f (M^n), then the Nijenhuis tensor N_f of f satisfying equation (1.5) is a tensor field of type (1, 2) given by [3]

$$N_f(X, Y) = [fX, fY] - f[f X, Y] - f[X, fY] + f^2[X, Y]. \qquad (6.1)$$

Let N^C be the Nijenhuis tensor of f^C in $T(M^n)$ of f in M^n, then we have

$$N^C(X^C, Y^C) = [f^C X^C, f^C Y^C] - f^C[f^C X^C, Y^C] - f^C[X^C, f^C Y^C] + (f^2)^C[X^C, Y^C]. \qquad (6.2)$$

For any X, Y (M^n) and f (M^n), we have

$$[X^C, Y^C] = [X, Y]^C \text{ and } [X + Y]^C = X^C + Y^C, \qquad (6.3)$$

$$f^C X^C = (f X)^C. \qquad (6.4)$$

From (1-8) and (6.4), we have

$$f^C m^C = (f m)^C = 0. \qquad (6.5)$$

Theorem 6.1. *The following identities hold,*

$$N^C(m^C X^C, m^C Y^C) = (f^C)^2[m^C X^C, m^C Y^C], \qquad (6.6)$$

$$m^C N^C(X^C, Y^C) = m^C[f^C X^C, f^C Y^C] \qquad (6.7)$$

$$m^C N^C(l^C X^C, l^C Y^C) = m^C[f^C X^C, f^C, Y^C] \qquad (6.8)$$

$$m^C N^C((f)^9 X^C, (f^C)^9 Y^C] = m^c[l^C X^C, l^C Y^C]. \qquad (6.9)$$

Proof. From equations (1.8), (1-9), (6.2) and (6.5) theorem can be proved easily.

Theorem 6.2. *The following identities hold.*

(i) $m^C N^C (X^C, Y^C) = 0,$

(ii) $m^C N^C (l^C X^C, l^C Y^C) = 0.$

(iii) $m^C N^C((f^C)^9 X^C,(f^C)^9 Y^C) = 0.$

Proof. In consequence of equation (6. 2), (1.8) and (1.9) it can be easily proved that $m^C N^C(l^C X^C, l^C Y^C) = 0$ if and only if $m^C N^C ((f^C)^9 X^C, (f^C)^9 Y^C) = 0$ for all X, Y (M^n)]. Now right hand side of the equations (6.7) and (6.8) are equal, which in view of equation (6.9) shows that above conditions are equivalent.

Theorem 6.3. *The complete lift of M^c of the. distribution M in $T(M^n)$ is integrable if and only if M is integrable in M^n.*

Proof. It is known that the distribution M is integrable in M^n if and only if

$$l[mX, mY] = 0, \quad \text{for any } X, Y \ (M^n). \qquad (6.10)$$

Taking complete lift of both sides, we get

$$l^C[m^C X^C, m^C Y^C] = 0, \qquad (6.11)$$

where $l^C = (m - I)^C = I - m^c$ is the projection tensor complementary to m^C. Thus, the conditions (6.10) and (6.11) are equivalent.

Theorem 6.4. *For any X, Y (M^n), let the distribution M be integrable in T(M^n) is integrable if and only if*

$$N(m X, mY) = 0.$$

Then the distribution M^c is integrable in $T(M^n)$ if and only if

$$l^C[m^C X^C, m^C Y^C] = 0$$

Or equivalently

$$N^C[m^C X^C, m^C Y^C] = 0.$$

Proof. By virtue of condition (6.6), we have

$$N^C(m^C X^C, m^C Y^C) = (f^C)^2[m^C X^C, m^C Y^C].$$

Multiplying throughout by l^C, we get

$$l^C N^C(m^C X^C, m^C Y^C) = (f^C)^2[m^C X^C, m^C Y^C],$$

which in view of (6.11) becomes

$$l^C N^C(m^C X^C, m^C Y^C) = 0. \qquad (6.12)$$

Also we have

$$m^C N^C(m^C X^C, m^C Y^C) = 0. \qquad (6.13)$$

Adding (6.10) and (6.13), we obtain

$$(l^C + m^C)\, N^C\, (m^C X^C, m^C Y^C] = 0,$$

since $l^C + m^C = I^C = I$, we have

$$N^C\,(m^C X^C, m^C Y^C) = 0.$$

Theorem 6.5. *For any X, Y (M^n), let the distribution M be integrable in M^n is integrable if and only if*

$$N(lX, lY) = 0.$$

Then the distribution L^C is integrable in $T(M^n)$ if and only if

$$m^C\,[l^C X^C, l^C Y^C]= 0.$$

or equivalently

$$N^C(l^C X^C, l^C Y^C) = 0.$$

Proof. Proof follows easily in a way similar to that of the Theorem 6.4.

Now, we define following

(i) . Distribution L is integrable

(ii) . Arbitrary vector field Z is tangent to an integral manifold of L.

(iii) . The operator f^*, such that $f^*Z = f\,Z$.

In view of equation (1.8) and (1.9) the induced structure f^* of f is an almost complex structure on each integral manifold L and f makes tangent spaces invariant of every integral manifold of L.

Definition 6.6. *The f-structure is partially integrable if the distribution L is integrable and the almost complex structure f^* induced from f on each integral manifold of L is also integrable.*

Let us denote the vector valued 2-form $N^*(Z,W)$ of the Nijenhuis tensor corresponding to the Nijenhuis tensor of the almost complex structure induced from f-structure on each integral manifold of L and for any $Z\ W\ (M^n)$ tangent to an integral manifold of L. Then we have

$$N(Z, W) = [f^* Z, f^*W] - f^*[f^* Z, W] - f^*[Z, f^*W] + f^{*2}\,[Z,W]. \qquad (6.14)$$

which in view of (6.2) and (6.12) yields

$$N^C(l^C X^C, l^C Y^C) = (N^*)^C(l^C X^C, l^C Y^C). \quad (6.15)$$

Theorem 6.7. *For any* X, *Y* (M^n), *let the f-structure he partially integrable i.e.,*

$$N(lX, lY) = 0.$$

Then the necessary and sufficient condition for f-structure to be partially integrable in $T(M^n)$ *is*

$$N^C(l^C X^C, l^C Y^C) = 0$$

Proof. In view of the equations (1.8); (1.9) (6.2), (6.15) and Theorem 6.5, the result follows easily.

When both the distributions *L* and M are integrable, we can choose a local coordinate system such that all *L* and *M* represented by putting $(n – r)$ local coordinates and r-coordinates constant respectively. We call such a coordinate system an adapted coordinate system. It can be supposed that in an adapted coordinate system the projection operator *I* and *m* have the components of the form

$$l = , \qquad m = ,$$

respectively. Where I_r denotes the unit matrix of order *r* and I_{n-r} is of order (n – r). Since *f* satisfies equation (1-8), the *f* has components of the form

$$f =$$

in an adapted coordinate system where f_r denotes *r r* square matrix.

Definition 6.8. *We say that an f-structure is integrable if:*

(i). *The structure f is partially integrable.*

(ii). *The distribution M is integrable i.e.,* $N(mX, mY) = 0$.

(iii). *The components of the f-structure are independent of the coordinates which are constant along the integral manifold of L in a adapted system.*

Theorem 6.9. *For any X, Y* (M^n), *let the f-structure be inte-grable in* M^n *if and only if*

$$N(X, Y) = 0$$

Then the necessary and sufficient condition for f-structure to be integrable in $T(M^n)$ *is*

$$N^C(X^C, Y^C)= 0.$$

Proof. In view of the equations (6.1) and (6.2), we get

Since *f*-structure is integrable in M^n. Therefore, the result follows.

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