The number of Smallest parts of *partitions* of n

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Abstract:

George E Andrews derived formula for the number of smallest parts of *partitions* of a positive integer *n*. In this paper we derived the formula efficiently by using the concepts of $r-partitions$ of n.

Keywords: *partition*, $r-partition$, smallest parts of the *partition* and $r-partition$ of positive integer *n*.

Subject classification: 11P81 Elementary theory of *partitions*.

Introduction

We adopt the common notation on *partitions* as used in [1]. A $partition$ of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, ..., \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$ and it is denoted by $n = (\lambda_1, \lambda_2, ..., \lambda_r)$. The λ_i are called the parts of the *partition*. Throughout this paper λ stands for a *partition* of n, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$, $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r$.

Let $\xi(n)$ denote the set of all *partitions* of n and $p(n)$ the cardinality of $\xi(n)$ for $n \in N$ and $p(0) = 1$. If $1 \leq r \leq n$ write $p_r(n)$ for the number of *partitions* of n each consisting of exactly r parts, i.e $r-partitions$ of *n*. If $r \leq 0$ or $r \geq n$ we write $p_r(n) = 0$. Let $p(k,n)$ represent the number of *partitions* of n using natural numbers at least as large as k only.

Let $spt(n)$ denote the number of smallest parts including repetitions in all *partitions* of n. Let us adopt the following notation. $m_s(\lambda) =$ number of smallest parts of λ.

$spt(n) = \sum_{\lambda \in \xi(n)} m_s(\lambda)$

1.1 Existing generating functions are given below.

Function	Generating function	
$p_r(n)$	$\frac{q^r}{(q)_r}$	
$p_r(n-k)$	$\frac{q^{r+k}}{(q)_r}$	
number of divisors	$\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)}$	
sum of divisors	$\sum_{n=1}^{\infty} \frac{n.q^n}{(1-q^n)}$	(1.1.1)

where $(q)_k = \prod_{n=1}^{k} (1-q^n)$ for $k > o$, $(q)_k = 1$ for $k = o$ and $(q)_k = 0$ for $k < o$.

Since $(a)_n = (a;q)_n = (1-a)(1-aq)(1-aq^2)...(1-aq^{n-1})$ [1]

1.2 Theorem: $spt(n) = \sum_{k=1}^{\infty} \sum_{t=1}^{\infty} p(k, n-tk) + d(n)$ (1.2.1)

Proof : [2] Let $n = (\lambda_1, \lambda_2, ..., \lambda_r) = (\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, ..., \mu_{l-1}^{\alpha_{l-1}}, k^{\alpha_l})$ be any $r-partition$ of n with *l* distinct parts.

Case 1: [3] Let $r > \alpha_l = t$ that means $\lambda_{r-t} > k$

Subtract all $k's$, we get $n - tk = (\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, ..., \mu_{l-1}^{\alpha_{l-1}})$

Hence $n - tk = (\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, ..., \mu_{l-1}^{\alpha_{l-1}})$ is a $(r-t)-partition$ of $n-tk$ with l - 1 distinct parts and each part greater than $k+1$. For corresponding to it they are $(r-t)-partitions$ of $n-tk$. Now we get, the number $p_{r-t}(k+1, n-tk)$ of $r-partitions$ of *n* with exactly t smallest elements as *k*.

Case 2: Let $r > \alpha_l > t$ that means $\lambda_{r-t} = k$

Omit $k's$ from last t places, we get $n-tk=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}},k^{\alpha_l-t}\right)$

Hence $n-tk=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},\ldots,\mu_{l-1}^{\alpha_{l-1}},k^{\alpha_l-t}\right)$ is a $(r-t)-partition$ of $n-tk$ with l distinct parts and the least part is k. For corresponding to it there are $r-partitions$ of $n-tk$ with least part k.

Now we get the number $f_{r-t}(k,n-tk)$ of $r-partitions$ of *n* with more than *t* smallest elements as *k*.

Case 3: Let $r=\alpha_l=t$ that means all parts in the *partition* are equal. For each $r-partition$ with equal parts have $r-partitions$ of *n*.

From cases (1), (2) and (3) we get $r-partitions$ of *n* with *t* smallest parts as *k* is

$$f_{r-t}(k,n-tk)+p_{r-t}(k+1,n-tk)+\beta$$

$$\text{where } \beta=1 \text{ if } r\,|\,n \text{ and } \beta=0 \text{ otherwise}$$

$$=f_{r-t}(k,n-tk)+p_{r-t}(k+1,n-tk)+\beta$$

$$=p_{r-t}(k,n-tk)+\beta$$

The number of *partitions* of *n* with equal parts is equal to the number of divisors of *n*. Since the number of divisors of *n* is $d(n)$. Then the number of $partitions$ of *n* with all parts are equal is $d(n)$.

From [5], the number of smallest parts in $partitions$ of *n* is

$$spt(n)=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}p(k,n-tk)+d(n)$$

1.3 Theorem: $p_r(k+1,n)=p_r(n-kr)$ (1.3.1)

Proof : Let $n=(\lambda_1,\lambda_2,\ldots,\lambda_r),\lambda_i>k\forall i$ be any $r-partition$ of n.

Subtracting each part by k, we get $n-kr=(\lambda_1-k,\lambda_2-k,\ldots,\lambda_r-k)$

Hence $n-kr=(\lambda_1-k,\lambda_2-k,\ldots,\lambda_r-k)$ is a $r-partition$ of $n-kr$.

Therefore the number of $r-partitions$ of n with parts greater than or equal to $k+1$ is $p_r(n-kr)$

1.4 Theorem: $\sum_{n=0}^{\infty}spt(n)q^n=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)}(q)_{n-1}$

Proof: From theorem (1.2.1), we have

$$spt(n)=\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}p(k,n-tk)+d(n)$$

Replace $k+1$ by k, n by $n-tk$ in (1.3.1)

$$=\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}p_r\left(n-tk-r\left(k-1\right)\right)+d(n)$$

Where $d(n)$ is the number of positive divisors of n.

From (1.1.1)

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{r+tk+r(k-1)}}{(q)_r}+\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{tk+rk}}{(q)_r}+\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}q^{tk}\left[\sum_{r=1}^{\infty}\frac{\left(q^k\right)^r}{(q)_r}\right]+\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\left[\left(1+\sum_{r=1}^{\infty}\frac{\left(q^k\right)^r}{(q)_r}\right)-1\right]+\sum_{r=1}^{\infty}\frac{q^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\left(1+\sum_{r=1}^{\infty}\frac{\left(q^k\right)^r}{(q)_r}\right)$$

$$=\sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\prod_{r=0}^{\infty}\left(\frac{1}{1-q^r q^k}\right) \qquad \text{from [1]}$$

$$=\sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\prod_{r=0}^{\infty}\left(\frac{1}{1-q^{r+k}}\right)$$

$$=\frac{1}{(q)_\infty}\sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}(q)_{k-1}$$

$$=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{q^n}{\left(1-q^n\right)}(q)_{n-1}$$

1.5 Corollary: The generating function for the number $A_c(n)$ of smallest parts of the *partitions* of n which are multiples of c is

$$\sum_{n=0}^{\infty} A_c(n)q^n = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{q^{cn}}{(1-q^{cn})}(q)_{cn-1}$$

1.6 Corollary: The generating function for the sum of smallest parts of the second *partitions* of n is

$$\sum_{n=0}^{\infty} sum\, spt(n)q^n = \frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{(1-q^n)}(q)_{n-1}$$

Proof: The generating function for the sum of smallest parts of the second *partitions* of a positive integer n is

$$spt(n) = \sum_{r=1}^{\infty}\sum_{t=1}^{\infty} k\ \ p(k, n-tk) + d(n)$$
$$= \sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty} k\ \ p_r\left(n-tk-r(k-1)\right) + d(n)$$

where $d(n)$ is the number of positive divisors of n.

From (1.1.1)

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{kq^{r+tk+r(k-1)}}{(q)_r} + \sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}$$

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{kq^{tk+rk}}{(q)_r} + \sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}$$

$$= \sum_{k=1}^{\infty}\sum_{t=1}^{\infty} kq^{tk}\left[\sum_{r=1}^{\infty}\frac{(q^k)^r}{(q)_r}\right] + \sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}$$

$$= \sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\left[\left(1+\sum_{r=1}^{\infty}\frac{(q^k)^r}{(q)_r}\right)-1\right] + \sum_{r=1}^{\infty}\frac{kq^k}{1-q^k}$$

$$= \sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\prod_{r=0}^{\infty}\left(\frac{1}{1-q^r q^k}\right)$$

$$= \sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\prod_{r=0}^{\infty}\left(\frac{1}{1-q^{r+k}}\right)$$

$$=\frac{1}{(q)_\infty}\sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}(q)_{k-1}$$

$$=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{(1-q^n)}(q)_{n-1}$$

$$\sum_{n=0}^{\infty} sum\,spt(n)q^n=\frac{1}{(q)_\infty}\sum_{n=1}^{\infty}\frac{nq^n}{(1-q^n)}(q)_{n-1}$$

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