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ON CONVEXITY OF WEIGHTED FUZZY MEAN DIVERGENCE MEASURES

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ABSTRACT
In the present communication, we define the weighted fuzzy mean Divergence Measures. Section 2 presents their functional forms, first and second derivatives, which are necessary for the study of convexity property of any measure. Section 3 presents the convexity and inequalities among fuzzy means divergence –measures.
Keywords : Weighted Fuzzy Mean of order t, weighted Fuzzy Arithmetic Mean, Geometric Mean, Harmonic Mean, Root-Square Mean, Convexity.
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1.1 INTRODUCTION

Means – Arithmetic, Geometric and Harmonic have their unique and fascinating role in algebraic studies. Recently studying probability distributions, for information theoretic, measures for increasing or decreasing probabilities, Singh and Pundir [1] have utilized arithmetic, geometric and harmonic means to characterize probability distributions have established new information theoretic measures. Such measures are very useful for the studies of population dynamics, budget planning and medical sciences, communication engineering and many others.

Complexity of life has given birth to uncertainties and fuzzy uncertainty is among them. Zadeh [6] introduced the concept of fuzzy set theory and there after a lot of research work has eased the complexity of life in this discipline. In the present study, our main objective is to discuss, fuzzy means divergence measures. Recently [2, 3] has considered some means analogous to information theoretic mean divergence measures studied by [4, 5]. Importance of event or experiment has been the outlook of every human being, therefore, we utilize the weighted distribution corresponding to fuzzy set theoretic distribution and consider the following fuzzy information scheme.

$$E_1 = \begin{bmatrix} E_1 & E_2 & .... & E_n \\ \mu_A(x_1) & \mu_A(x_2) & .... & \mu_A(x_n) \\ w_1 & w_2 & .... & w_n \end{bmatrix} \tag{1.1}$$

Hence the fuzzy weighted entropy is given by :

$$F(\mu_A(x_i); w_i) = -w_i[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A x_i))\log(1-\mu_A(x_i))] \tag{1.2}$$

Since the basic aim is to study the fuzzy mean divergence measures so we consider the following fuzzy information scheme:

$$E_2 = \begin{bmatrix} E_1 & E_2 & E_3 & ... & E_v \\ \mu_A(x_1) & \mu_A(x_2) & \mu_A(x_3) & ... & \mu_A(x_n) \\ \mu_B(x_1) & \mu_B(x_2) & \mu_B(x_3) & ... & \mu_B(x_m) \\ w_1 & w_2 & w_3 & ... & w_n \end{bmatrix} \tag{1.3}$$

where

$$E = [E_1,\ E_2, ..., E_n]$$
$$A = [\mu_A(x_i);\quad i = 1, 2, ..., n]$$
$$B = [\mu_B(x_i);\quad i = 1, 2, ..., n]$$
$$W = [w_i(x_i);\quad i = 1, 2, ..., n]$$

Taking into consideration (1.3), we have fuzzy weighted divergence as :

$$F(A \| B; W) = \sum_{i=1}^{n} w_i \left[\mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_B(x_i)} + (1 - \mu_A(x_i)) \log \frac{(1 - \mu_A(x_i))}{1 - \mu_B(x_i))} \right] \tag{1.4}$$

Since the main objective of this study is the Fuzzy weighted mean divergence measures, so we consider the generalized weighted fuzzy mean divergence of order t, t ≠ 0 as follows:

$$M_t(\mu_A;\ \mu_B; W) = w_i \left[\left\{ \frac{\mu_A^t(x_i) + \mu_B^t(x_i)}{2} \right\}^{\frac{1}{t}} + \left\{ \frac{(1 - \mu_A(x_i))^t + (1 - \mu_B(x_i))^t}{2} \right\}^{1/t} \right]$$

where t ≠ 0

Particular cases

1. Harmonic Mean =

$$M_{-1}(\mu_A, \mu_B; W) = H(\mu_A(x_i), \mu_B(x_i); W)$$
$$= w_i \left\{ \frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A(x_i) + \mu_B(x_i)} + \frac{2(1 - \mu_A(x_i))(1 - \mu_B(x_i))}{2 - \mu_A(x_i) - \mu_B(x_i)} \right\}$$

where t = –1

2. Geometric Mean =

$$M_0(\mu_A,\ \mu_B; W) = G(\mu_A(x_i), \mu_B(x_i); W)$$
$$= w_i \left\{ \sqrt{\mu_A(x_i)\mu_B(x_i)} + \sqrt{(1 - \mu_A(x_i)(1 - \mu_B(x_i)} \right\}$$

Where t = 0

3. Arithmetic Mean =

$$M_1(\mu_A, \mu_B; W) = A(\mu_A(x_i), \mu_B(x_i); w)$$
$$= w_i \left\{ \frac{\mu_A(x_i) + \mu_B(x_i)}{2} + \frac{2 - \mu_A(x_i) - \mu_B(x_i)}{2} \right\}$$

where t = 1

4. Root-Square Mean =

$$M_2(\mu_A,\ \mu_B; W) = S(\mu_A(x_i), \mu_B(x_i); w)$$
$$= w_i \left\{ \sqrt{\frac{\mu_A^2(x_i) + \mu_b^2(x_i)}{2}} + \sqrt{\frac{1 - \mu_A(x_i)(1 - \mu_B(x_i)}{2}} \right\}$$

Where t = 2

5. $M_{1/2}(\mu_A, \mu_B; W) = H_1(\mu_A(x_i), \mu_B(x_i); w)$

$$= w_i \left\{ \left(\frac{\sqrt{\mu_A(x_i)} + \sqrt{\mu_B(x_i)}}{2} \right)^2 \left(\frac{\sqrt{1-\mu_A(x_i)} + \sqrt{(1-\mu_B(x_i)}}{2} \right)^2 \right\}$$

when t = 1/2

6. $M_\infty(\mu_A, \mu_B, W) = \max(\mu_A(x_i), \mu_B(x_i); w)$, t = ∞

7. $M_{-\infty}(\mu_A,\ \mu_B; W),\ \min(\mu_A(x_i), \mu_B(x_i); w)$, t = –∞

MIXED WEIGHTED FUZZY MEAN MEASURES

1. $N_2(\mu_A(x), \mu_B(x); W) = \sqrt{N_1(\mu_A(x),\ \mu_B(x); W.A(\mu_A(x),\ \mu_B(x); W)}$

$$= w_r \left[\left\{ \left(\frac{\sqrt{\mu_A(x)} + \sqrt{\mu_B(x)}}{2} \right)^2 + \left(\frac{\sqrt{1-\mu_A(x)} + \sqrt{1-\mu_B(x)}}{2} \right)^2 \right\} \left\{ \frac{\mu_A(x_i) + \mu_B(x_i)}{2} + \frac{2-\mu_A(x_i) - \mu_B(x_i)}{2} \right\} \right]$$

or

$$= w \left[\left\{ \frac{\sqrt{\mu_A}(x) + \sqrt{\mu_B}(x)}{2} \right\} \left\{ \frac{\sqrt{\mu_A(x) + \mu_B(x)}}{2} \right\} + \left\{ \frac{\sqrt{1-\mu_A(x)} + \sqrt{1-\mu_B(x)}}{2} \right\} \left\{ \frac{\sqrt{2-\mu_A(x) - \mu_B(x)}}{2} \right\} \right]$$

2. $N_3(\mu_A x), \mu_B(x); W) = \dfrac{2A(\mu_A(x), \mu_B(x); w) + G(\mu_A(x),\ \mu_B(x); w)}{3}$

$$= w \left[\frac{\{(\mu_A(x_i) + \mu_b(x_i) + (1-\mu_A(x_i)(1-\mu_B(x_i)\}}{3} + \left\{ \frac{\sqrt{\mu_A(x_i) + \mu_B(x_i) + (1-\mu_A(x_i)(1-\mu_B(x_i)}}{3} \right\} \right]$$

or

$$= w \left[\left\{ \frac{(\mu_A(x) + \mu_B(x) + \sqrt{\mu_A(x) + \mu_B(x)}}{3} \right\} + \left\{ \frac{1-\mu_A(x) - \mu_B(x) + \sqrt{(1-\mu_A(x))(1-\mu_B(x))}}{3} \right\} \right]$$

In the next section, we present the weighted fuzzy mean– divergence measures.

SECTION-2

2.1 WEIGHTED FUZZY MEAN DIVERGENCE MEASURES

The main concern of this study is to present the concavity of measures, so we define the weighted fuzzy mean difference measures as follows:

S.No.	Measure	Expression
1.	Weighted Fuzzy Root square Arithmetic Mean divergences measure	$M_{SA}(A\|\|B;W)=\sum_{i=1}^{n} w_i\left[\sqrt{\frac{\mu_A^2(x_i)+\mu_B^2(x_i)}{2}}+\sqrt{\frac{(1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2}{2}}-\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2}\right)-\frac{2-\mu_A(x_i)-\mu_B(x_i)}{2}\right]$
2.	Weighted Fuzzy Root square Geometric Mean Divergence Measure	$M_{SG}(A\|\|B;w)=\sum_{i=1}^{n} w_i\left[\sqrt{\frac{(\mu_A(x_i))^2+(\mu_B(x_i))^2}{2}}+\sqrt{\frac{(1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2}{2}}-\sqrt{\mu_A(x_i)+\mu_B(x_i)}-\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}\right]$
3.	Weighted Fuzzy Square Root-Harmonic Mean Divergence	$M_{SH}(A\|\|B;\ W)=\sum_{i=1}^{n} w_i\left[\sqrt{\frac{(\mu_A(x_i))^2+(\mu_B(x_i))^2}{2}}+\sqrt{\frac{(1-\mu_A(x_i))^2+(1-\mu_B(x_i))^2}{2}}-\frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}-\frac{2(1-\mu_A(x_i))(1-\mu_B(x_i))}{2-\mu_A(x_i)-\mu_B(x_i)}\right]$
4.	Weighted Arithmetic Geometric Mean Divergence Measure	$M_{AG}(A\|\|B;\ W)=\sum_{i=1}^{n} w_i\left[\frac{\mu_A(x_i)+\mu_B(x_i)}{2}+\frac{2-\mu_A(x_i)-\mu_B(x_i)}{2}-\sqrt{\mu_A(x_i)\mu_B(x_i)}-\sqrt{(1-\mu_A(x_i)(1-\mu_B(x_i))}\right]$

5.	Weighted Arithmetic – Harmonic Mean Divergence Meausre	$M_{AH}(A\|B;\ W)=\sum_{i=1}^{n} w_i\left[\frac{\mu_A(x_i)+\mu_B(x_i)}{2}+\frac{2-\mu_A(x_i)-\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}-\frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}-\frac{2(1-\mu_A(x_i))(1-\mu_B(x_i))}{2-\mu_A(x_i)-\mu_B(x_i)}\right]$
6.	Weighted Fuzzy Geometric –Haronic Mean Divergence Measure	$M_{GH}(A\|B;\ W)=\sum_{i=1}^{n} w_i\left[\sqrt{\mu_A(x_i)\mu_B(x_i)}+\sqrt{(1-\mu_A(x_i))(1-\mu_B(x_i))}-\frac{2\mu_A(x_i)\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}-\frac{2(1-\mu_A(x_i))(1-\mu_B(x_i))}{2-\mu_A(x_i)-\mu_B(x_i)}\right]$
7.	Weighted Fuzzy $N_2 - N_1$ Mean Divergence Measure	$M_{N_2N_1}(A\|B;\ W)=\sum_{i=1}^{n} w_i\left[\sqrt{\mu_A(x_i)}+\sqrt{\mu_B(x_i)}\times\times\frac{\sqrt{\mu_A(x_i)\mu_B(x_i)}}{2}+\left(\frac{\sqrt{1-\mu_A(x_i)}+\sqrt{1-\mu_B(x_i)}}{2}\right)\times\left(\frac{\sqrt{(1-\mu_A(x_i))+(1-\mu_B(x_i))}}{2}\right)-\left(\frac{\sqrt{\mu_A(x_i)}+\sqrt{\mu_B(x_i)}}{2}\right)^2\times\left(\frac{\sqrt{(1-\mu_A(x_i))}+\sqrt{(1-\mu_B(x_i))}}{2}\right)^2\right]$

8.	Weighted Fuzzy N_2 – Geometric Mean Divergence Measure	$M_{N_2G}(A\|B;\ W)=\sum_{i=1}^{n} w_i\left[\frac{\sqrt{\mu_A(x_i)}+\sqrt{\mu_B(x_i)}}{2}\times\frac{\sqrt{\mu_A(x_i)+\mu_B(x_i)}}{2}+\frac{\sqrt{1-\mu_A(x_i)}+\sqrt{1-\mu_B(x_i)}}{2}\times\frac{\sqrt{2-\mu_A(x_i)-\mu_B(x_i)}}{2}-\sqrt{\mu_A(x_i)\mu_B(x_i)}-\sqrt{(1-\mu_A(x_i)(1-\mu_B(x_i))}\right]$
9.	Weighted Fuzzy Arithmetic – N_2 Mean Divergence Measure	$M_{AN_2}(A\|B;\ W)=\sum_{i=1}^{n} w_i\left[\frac{\mu_A(x_i)+\mu_B(x_i)}{2}+\frac{2-\mu_A(x_i)-\mu_B(x_i)}{2}-\left(\frac{\sqrt{\mu_A(x_i)}+\sqrt{\mu_B(x_i)}}{2}\right)\times\left(\frac{\sqrt{\mu_A(x_i)+\mu_B(x_i)}}{2}\right)-\left(\frac{\sqrt{1-\mu_A(x_i)}+\sqrt{1-\mu_B(x_i)}}{2}\right)\times\left(\frac{\sqrt{2-\mu_A(x_i)-\mu_B(x_i)}}{2}\right)\right]$

SECTION 3

3.0 CONVEXITY OF WEIGHTED FUZZY MEAN DIVERGENCE MEASURES

In fact, properties like convexity and concavity are very much important from analysis point of view. The convexity may lead to minimization of the measure under certain conditions in a prescribed domain. In case of uncertainties of any kind – probabilistic or fuzzy, the domain is [0, 1]. So in this section we obtain in first and second derivatives to study the convexity of the weighted fuzzy mean divergences defined earlier. Setting $\mu_A(x_i)=x$ and $\mu_B(a_i)=\frac{1}{2}$, we have the following functional forms their first and second derivatives and discuss the convexity through them.

3.1 Convexity of Weighted fuzzy Root Square– Arithmetic Mean Divergence Measure

We have

$$M_{SA}(A\|B;W)=w\left\{\frac{\sqrt{x^2+\frac{1}{4}}}{2}+\sqrt{\frac{(1-x)^2+\frac{1}{4}}{2}}-1\right\}, \tag{3.1.1}$$

after setting $\mu_A(x_0)=x,\ \mu_B(x_i)=\frac{1}{2}$

$$= f_{SA}(x;w)$$

$$f'_{SA}(x;w)=w\left\{\frac{x}{2}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}-\frac{(1-x)}{2}\left(\frac{(1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}\right\} \tag{3.1.2}$$

$$f''_{SA}(x;w)=w\left\{\frac{1}{2}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}-\frac{x^2}{4}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{3}{2}}\right\}$$

$$+\frac{1}{2}\left(\frac{1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}\frac{(1-x)^2}{4}\left[\frac{(1-x)^2+\frac{1}{4}}{2}\right]^{-\frac{3}{2}}. \tag{3.1.3}$$

Since $f''_{SA}(x;w)$ > 0, ∀ x ∈ (0, 1), w > 0, hence the function f_{SA}(x; w) is *convex and non-negative* ∀ x ∈ (0, 1), w > 0 and $f\left(\frac{1}{2};1\right)=0$.,

Since

, at $x=\frac{1}{2}$ and $f''_{SA}(x;w)$ > 0, ∀ x ∈ (0, 1), w > 0, hence minimum at $x=\frac{1}{2}$.

3.2 Convexity of Weighted Fuzzy Root Square– Geometric Mean Divergence

We have

$$M_{SG}(A\|B;\ W)=f_{SG}(x;w)\ \mu_A(x)=x,\quad \text{after setting}\ \ \mu_B(x_i)=\frac{1}{2},$$

$$=w\left[\frac{\sqrt{x^2+\frac{1}{4}}}{2}+\frac{\sqrt{(1-x)^2+\frac{1}{4}}}{2}-\sqrt{\frac{x}{2}}-\sqrt{\frac{1-x}{2}}\right] \tag{3.1.4}$$

So,

$$f'_{SG}(x;w)=w\left[\frac{x}{2}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}-\frac{(1-x)}{2}\left\{\frac{(1-x)^2+\frac{1}{4}}{2}\right\}^{-\frac{1}{2}}-\frac{1}{4}\left(\frac{x}{2}\right)^{-\frac{1}{2}}+\frac{1}{4}\left(\frac{1-x}{2}\right)^{-\frac{1}{2}}\right] \quad (3.1.5)$$

and

$$f''_{SG}(x;w)=w\left[\frac{1}{2}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}-\frac{x^2}{4}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{3}{2}}+\frac{1}{2}\left\{\frac{(1-x)^2+\frac{1}{4}}{2}\right\}^{-\frac{1}{2}}\right.$$

$$\left.-\frac{(1-x)^2}{4}-\left\{\frac{(1-x)^2+\frac{1}{2}}{2}\right\}^{-\frac{3}{2}}+\frac{1}{16}\left(\frac{x}{2}\right)^{-\frac{3}{2}}+\frac{1}{16}\left(\frac{1-x}{2}\right)^{-\frac{3}{2}}\right]. \quad (3.1.6)$$

We observe that $f''_{SA}(x;w) > 0 \ \forall \ x \in (0, 1), w > 0 \Rightarrow f_{SG}(x, w))$ is minimum at $x=\frac{1}{2}$ and *convex and non-negative* in $x \in (0,1)$, $w > 0$.

3.3 Convexity of Fuzzy Root Square Harmonic Mean Divergence - Measure

We have

$M_{SH}(A\|B;\ W)=f_{SH}(x;w)$, after setting

$\mu_A(x_i)=x, \quad \mu_B(x_i)=\frac{1}{2}$ i.e.

$$f_{SH}(x;\ w)=w\left[\frac{\sqrt{x^2+\frac{1}{4}}}{2}+\frac{\sqrt{(1-x)^2}+\frac{1}{4}}{2}-\frac{x}{x+\frac{1}{2}}-\frac{1-x}{\frac{3}{2}-x}\right] \quad (3.1.7)$$

$$f'_{SH}(x;\ w)=w\left[\frac{x}{2}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}\frac{(1-x)}{2}\left\{\frac{(1-x)^2+\frac{1}{4}}{2}\right\}^{-\frac{1}{2}}-\frac{1}{2\left(x+\frac{1}{2}\right)}+\frac{1}{2\left(\frac{3}{2}-x\right)}\right] \quad (3.1.8)$$

and

$$f''_{SH}(x;\ w)=w\left[\frac{1}{2}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}-\frac{x^2}{4}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{3}{2}}+\frac{1}{2}\left\{\frac{(1-x)^2+\frac{1}{4}}{2}\right\}^{-\frac{1}{2}}\right.$$

$$-\frac{(1-x)^2}{4}\left\{\frac{(1-x)^2+\frac{1}{4}}{2}\right\}^{-\frac{3}{2}}+\frac{1}{\left(x+\frac{1}{2}\right)^3}+\frac{1}{\left(\frac{3}{2}-x\right)^3}\Bigg] \tag{3.1.9}$$

We observe from (3.1.9) that $f''_{SH}(x;w)>0$, $\forall$ x $\in$ (0, 1), w > 0, hence $f_{SH}(x;\, w)$ is *Convex and Non-Negative* in x $\in$ (0, 1), w > 0.

3.4 Convexity of Weighted Fuzzy Arithmetic-Geometric Mean Divergence - Measure

We have

$M_{AG}(A\|B;W)$ after setting $\mu_A(x_i) = x$, $\mu_B(x_i) = \frac{1}{2}$

$$f_{AG}(x;\, w)=w\left[1-\sqrt{\frac{x}{2}}-\sqrt{\frac{1-x}{2}}\right] \tag{3.1.10}$$

$$f'_{AG}(x;\, w)=w\left[-\frac{1}{4}\left(\frac{x}{2}\right)^{-\frac{1}{2}}+\frac{1}{4}\left(\frac{1-x}{2}\right)^{-\frac{1}{2}}\right] \tag{3.1.11}$$

and

$$f''_{AG}(x;\, w)=\frac{w}{16}\left[\left(\frac{x}{2}\right)^{-\frac{1}{2}}-\left(\frac{1-x}{2}\right)^{-\frac{3}{2}}\right]. \tag{3.1.12}$$

From (3.1.12), we conclude that $f''_{AG}(x;w)>0$, $\Rightarrow$ that $f_{AG}(x;\, w)$ is *Convex* and *Non-Negative* for all $x \in (0, 1)$, $w > 0$.

3.5 Convexity of Weighted Fuzzy Arithmetic Geometric Mean Divergence - Measure

We have

$M_{AH}(A\|B;W)$ after setting $\mu_A(x_i) = x$, $\mu_B(x_i) = \frac{1}{2}$

$$f_{AH}(x;\, w)=w\left[1-\frac{x}{x+\frac{1}{2}}-\frac{1-x}{\frac{3}{2}-x}\right] \tag{3.1.13}$$

$$\Rightarrow \quad f'_{AH}(x;\, w)=\frac{w}{2}\left[-\frac{1}{\left(x+\frac{1}{2}\right)^2}+\frac{1}{\left(\frac{3}{2}-x\right)^2}\right] \tag{3.1.14}$$

and

$$f''_{AH}(x;w)=w\left[\frac{1}{\left(x+\frac{1}{2}\right)^3}+\frac{1}{\left(\frac{3}{2}-x\right)^3}\right] \tag{3.1.15}$$

From (3.1.15), we conclude that $f''_{AH}(x;w)>0$, $\forall$ x $\in$ (0, 1), $w > 0$, hence $f_{AH}(x;\, w)$ is *Convex* and *Non-Negative* in $x \in (0, 1)$, w > 0.

3.6 Convexity of Weighted Fuzzy N₂ – N₁ Mean Divergence - Measure

$M_{N_2N_1}(A\|B;W)$ after setting $\mu_A(x_i) = x$, and $\mu_B(x_i) = \frac{1}{2}$

$$f_{N_2N_1}(x;w) = w\left[\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\sqrt{\frac{x+\frac{1}{2}}{2}}\right)+\left(\sqrt{\frac{1-x}{2}}+\sqrt{\frac{1}{2}}\right)\times\frac{\sqrt{\frac{3}{2}-x}}{2}-\left(\frac{\sqrt{x}+\frac{1}{\sqrt{2}}}{2}\right)^2-\left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{2}\right)^2\right]. \quad (3.1.16)$$

$$\Rightarrow \quad f'_{N_2N_1}(x;w) = \frac{1}{4\sqrt{x}}w\left[\sqrt{\frac{x+\frac{1}{2}}{2}}+\frac{1}{4}\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\frac{x+\frac{1}{2}}{2}\right)^{-\frac{1}{2}} - \frac{1}{4\sqrt{1-x}}\left(\frac{\sqrt{\frac{3}{2}-x}}{2}\right)-\frac{1}{4}\left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\frac{\frac{3}{2}-x}{2}\right)^{-\frac{1}{2}} - \frac{1}{4\sqrt{2x}}+\frac{1}{4\sqrt{2(1-x)}}\right]. \quad (3.1.17)$$

and

$$f''_{N_2N_1}(x;w) = -w\left[\left\{\frac{x\frac{\sqrt{x}}{2}+\frac{1}{4}}{8\sqrt{2}\,x^{3/2}\left(x+\frac{1}{2}\right)^{\frac{3}{2}}}+\frac{(1-x)\sqrt{\frac{1-x}{2}}+\frac{1}{4}}{8\sqrt{2}\,(1-x)\left(\frac{3}{2}-x\right)^{\frac{3}{2}}}\right\}-\frac{1}{4(2x)^{3/2}}\frac{1}{4(2(1-x))^{3/2}}\right] \quad (3.1.18)$$

From (3.1.18), we conclude that $f''_{N_2N_1}(x;w) > 0$, $\forall\ x \in (0, 1)$, $w > 0$. Hence $f_{N_2N_1}(x;w)$ is *Convex and Non-Negative* in $x \in (0, 1)$, $w > 0$.

3.7 Convexity of Weighted Fuzzy N₂ – Geometric Mean Divergence - Measure

$M_{N_2G}(A\|B;W)$ after setting $\mu_A(x_i) = x$, and $\mu_B(x_i) = \frac{1}{2}$

$$f_{N_2G}(x;w) = w\left[\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\sqrt{\frac{x+\frac{1}{2}}{2}}\right)+\left(\frac{\sqrt{\frac{1-x}{2}}+\sqrt{\frac{1}{2}}}{2}\right)\times\left(\sqrt{\frac{\frac{3}{2}-x}{2}}\right)-\sqrt{\frac{x}{2}}-\sqrt{\frac{1-x}{2}}\right] \quad (3.1.19)$$

$$\Rightarrow \quad f'_{N_2G}(x;w) = w\left[\frac{1}{4\sqrt{x}}\sqrt{\frac{x+\frac{1}{2}}{2}} + \left(\frac{\sqrt{x}+\frac{1}{\sqrt{2}}}{8}\right)\left(\frac{x+\frac{1}{2}}{2}\right)^{-\frac{1}{2}} - \frac{1}{4\sqrt{1-x}}\left(\frac{\sqrt{\frac{3}{2}-x}}{2}\right) - \left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{8}\right)\left(\frac{\frac{3}{2}-x}{2}\right)^{-\frac{1}{2}} - \frac{1}{4}\left(\frac{x}{2}\right)^{-\frac{1}{2}} + \frac{1}{4}\left(\frac{1-x}{2}\right)^{-\frac{1}{2}}\right] \quad (3.1.20)$$

and

$$f''_{N_2G}(x;w) = w\left[Q + \frac{1}{16}\left(\frac{x}{2}\right)^{-3/2} + \frac{1}{16}\left(\frac{1-x}{2}\right)^{-3/2}\right] \quad (3.1.21)$$

where

$$Q = -\left\{\frac{x\frac{\sqrt{x}}{2}+\frac{1}{4}}{8\sqrt{2}\,x^{3/2}\left(x+\frac{1}{2}\right)^{\frac{3}{2}}} + \frac{(1-x)\sqrt{\frac{1-x}{2}}+\frac{1}{4}}{8\sqrt{2}\,(1-x)\left(\frac{3}{2}-x\right)^{\frac{3}{2}}}\right\} \quad (3.1.22)$$

We observe from (3.1.21) (3.1.22) that $f''_{N_2G}(x;w) > 0$, $\forall\ x \in (0, 1)$, $w > 0$. Hence $f_{N_2G}(x;w)$ is *Convex* and *Non-Negative* in $\forall\ x \in (0, 1)$, $w > 0$.

3.8 Convexity of N_2 – Geometric Mean Divergence Measure

We have

$M_{N_2G}(A\|B;w) = f_{N_2G}(z;w)$ after setting $\mu_A(x_i) = x$, and $\mu_B(x_i) = \frac{1}{2}$

$$f_{N_2G}(x;w) = w\left[\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\frac{\sqrt{x+\frac{1}{2}}}{2}\right) + \left(\frac{\sqrt{\frac{1-x}{2}}+\sqrt{\frac{1}{2}}}{2}\right) \times \left(\frac{\sqrt{\frac{3}{2}-x}}{2}\right) - \sqrt{\frac{x}{2}} - \sqrt{\frac{1-x}{2}}\right] \quad (3.1.23)$$

$$\Rightarrow \quad f'_{N_2G}(x;w) = w\left[\frac{1}{4\sqrt{x}}\sqrt{\frac{x+\frac{1}{2}}{2}} + \left(\frac{\sqrt{x}+\frac{1}{\sqrt{2}}}{8}\right)\left(\frac{x+\frac{1}{2}}{2}\right)^{-\frac{1}{2}} - \frac{1}{4\sqrt{1-x}}\left(\frac{\sqrt{\frac{3}{2}-x}}{2}\right) - \left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{8}\right)\left(\frac{\frac{3}{2}-x}{2}\right)^{-\frac{1}{2}} - \frac{1}{4}\left(\frac{x}{2}\right)^{-\frac{1}{2}} + \frac{1}{4}\left(\frac{1-x}{2}\right)^{-\frac{1}{2}}\right] \quad (3.1.24)$$

and

$$f''_{N_2G}(x;w) = w\left[Q + \frac{1}{16}\left(\frac{x}{2}\right)^{-3/2} + \frac{1}{16}\left(\frac{1-x}{2}\right)^{-3/2}\right] \tag{3.1.25}$$

where Q is given by (3.1.22).

From (3.1.25), we conclude that $f''_{N_2G}(x;w) > 0$, $\forall\ x \in (0, 1)$, $w > 0$. Hence $f_{N_2G}(x;w)$ is *Convex* and *Non-Negative* in $\forall\ x \in (0, 1)$, $w > 0$.

3.9 Convexity of Weighted Fuzzy Arithmetic-N_2 Mean Divergence Measure

We have

$M_{AN_2}(A\|B;w)$ after setting $\mu_A(x_i) = x$, and $\mu_B(x_i) = \frac{1}{2}$

$$f_{AN_2}(x;w) = w\left[1 - \left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\frac{\sqrt{x+\frac{1}{2}}}{2}\right) + \left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\frac{\sqrt{\frac{3}{2}-x}}{2}\right)\right] \tag{3.1.26}$$

$$\Rightarrow \quad f'_{AN_2}(x;w) = w\left[-\frac{1}{4\sqrt{x}}\sqrt{\frac{x+\frac{1}{2}}{2}} - \frac{1}{4}\left(\frac{\sqrt{x}+\frac{1}{\sqrt{2}}}{2}\right)\left(\frac{x+\frac{1}{2}}{2}\right)^{-\frac{1}{2}}\right.$$

$$\left. + \frac{1}{4\sqrt{1-x}}\left(\frac{\sqrt{\frac{3}{2}-x}}{2}\right) + \frac{1}{4}\left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\frac{\frac{3}{2}-x}{2}\right)^{-\frac{1}{2}}\right]. \tag{3.1.27}$$

and

$$f''_{AN_2}(z;w) = -Qw \tag{3.1.28}$$

But $f''_{AN_2}(z;w) > 0$, as Q is –ive $\forall\ x \in (0, 1)$. Hence $f_{AN_2}(z;w)$ is *convex and non-negative* in $x \in (0, 1)$, $w > 0$.

CONCLUSION

From above discussion, it is very clear that each Weighted Fuzzy Mean Divergence Measure is *Convex* and *Non-Negative* $\forall\ x \in (0, 1)$, $w > 0$.

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