A view on g -Fuzzy Compactness

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Abstract:

In this paper, we give a brief introduction of g-Fuzzy Topological Spaces using the two additive operations sum ⊕ and conjunction &. Our objective is to introduce compactness in g- fuzzy topological spaces. Also, we wish to obtain some properties of g-fuzzy compact Spaces.

Key words:

Fuzzy sets, fuzzy operations, g- fuzzy topological spaces, g- fuzzy Compactness

Introduction

The fundamental definitions and theorems of all branches of Mathematics, with respect to ordinary sets are considered as a particular case of the concept of fuzzy set. Several ordinary results can be generalized. Fuzziness is one of the most important and useful concepts in the modern theories of scientific studies. Zadeh in 1965 introduced the concept of fuzzy set. The fuzzy topological space was introduced by Chang in 1968 and Lowen in 1976. Since then various notions in classical topology have been extended. A number of research works have been dedicated on development of various aspects of the theory and applications of fuzzy sets. In recent years, fuzzy topology has developed considerably. Mathews and Samuel in 2008 introduced an alternate and more general definition of fuzzy topological spaces called g-fuzzy topological spaces using the two additive operations sum ⊕ and conjunction &.

1. Basic Concepts

We give a brief account of the developments right from fuzzy sets and g-fuzzy topological spaces up to g- fuzzy compactness. We begin with definitions.

Definition 2.1 Let X be a non empty set. A fuzzy set A in X is characterized by a membership function $\mu : X \rightarrow [0,1]$ where [0,1] is the closed unit interval, while an ordinary set $A \subseteq X$ is identified with its characteristic function $\chi_A: X \rightarrow \{0, 1\}$.

We use the same symbols, capital letters A, B, C… to denote both fuzzy sets and sets in classical set theory. Membership functions of fuzzy sets A, B, C… are denoted by μ_A, μ_B, μ_C. If A denotes a fuzzy set, $\mu_A(x)$ is called the grade of membership of x in A.

Let I(X) be the family of all the fuzzy sets in X called fuzzy space and P(X) be the class of fuzzy sets whose membership functions have all their values in {0,1}. An ordinary set 'A' in P(X) can be identified by its characteristic function $\chi_A: X \rightarrow \{0, 1\}$, it is a special fuzzy set. Therefore, $P(X) \subset I(X)$.

The fuzzy sets Φ and X are given by Φ(x) = 0 and X(x) = 1 for all $x \in X$.

In I (X) the following additive operations can be introduced:

Definition 2.2

i. The sum of two fuzzy sets A and B in a set X ,denoted by $A \oplus B$,is a fuzzy set in X defined by $(A \oplus B)(x) = \min(1, A(x) + B(x))$ for all $x \in X$

ii. The difference of two fuzzy sets A and B in X ,denoted by $A \Theta B$ is a fuzzy set in X defined by $(A \Theta B)(x) = \max(0, A(x) - B(x))$ for all $x \in X$

iii. The conjunction of two fuzzy sets A and B ,denoted by A&B , is a fuzzy set in X defined by $(A\&B)(x) = \max(0, A(x)+B(x)-1)$ for all $x \in X$

iv. The product A.B is a fuzzy set defined by
 (A.B)(x) = A(x).B(x) for all $x \in X$.

3. g- Fuzzy Topological Spaces

We have introduced an alternate and more general definition of fuzzy topological spaces called g-fuzzy topological space or g-fts. The sum $\oplus$ and conjunction & for an indexed family of fuzzy sets can be defined as follows:

Definition 3.1

Let J be an infinite index set and $J_i \subset J$ be finite or countable set.

Similar to operations on ordinary sets, we can generalize the *sum* of any family $\{A_i \ / \ i \in J\}$ of fuzzy sets of a non empty set X as $(\oplus_{i\in J} A_i)(x) = \sup_{Ji \subset J} ((\oplus_{i\in Ji} A_i)(x)$ for $x \in X$

In a similar way we define the *conjunction* of any family $\{\mu_i \ / \ I \in J\}$ of fuzzy sets of a non empty set X as

$$(\&_{i\in J} A_i)(x) = \inf_{Ji \subset J} ((\&_{i\in Ji} A_i)(x) \text{ for } x \in X$$

Definition 3.2 Let X be a non empty set. A family δ of fuzzy sets in X is called a g-fuzzy topology on X if

1. $X \in T$ and $\Phi \in \delta$
2. $A \,\&\, B \in \delta$ whenever $A, B \in \delta$ and
3. $(\oplus_\alpha A_\alpha) \in \delta$ for any subfamily $\{A_\alpha\}_{\alpha\in J}$ in δ

The set X together with a g- fuzzy topology δ , denoted by (X, δ) is called a g-fuzzy topological space. Members of δ are called g- open fuzzy sets in X. The complement of a g- open fuzzy set is called g-closed fuzzy set.

Example 3.3 Let X = {a, b, c} and A = {(a, 0), (b, 0.5), (c, 1)} be a fuzzy subset of X. Then, δ = {Φ, X, A} is a g-fuzzy topology on X and corresponding g- fuzzy topological space (X, δ) is a g- fuzzy topological space.

Example 3.4 Let X = { a, b, c, d }.Define A , B , C : X → [0, 1] by

A (x) = 1 if x = a and 0 elsewhere

B (x) = 1 if x = b and 0 elsewhere

C (x) = 1 if x = a, b and 0 elsewhere

Let δ = { $\underline{0}$, $\underline{1}$, A , B, C }.Then, (X , δ) is a g- fuzzy topological space.

Definition 3.5 Let (X,δ) and (Y,η) be two g-fuzzy topological spaces and f : X → Y be a function. Then f is said to be a g-fuzzy continuous function if $f^{-1}(\mu) \in \delta$ for each $\mu \in \eta$.

Definition 3.6 A gfts (X,δ) is said to have the Hausdorff property or to be a Hausdorff if for each pair x , y $\in$ X with x $\neq$ y , implies that there exist fuzzy open sets μ and ν with μ (x) = $\underline{1}$ = ν (y) and μ & ν = $\underline{0}$.

4. G-fuzzy Compact Spaces and their properties

Now we introduce the concepts of open cover and compactness in g - fuzzy topological spaces to get more general results on fuzzy compactness

Definition 4.1 A collection $\{\mu_\alpha\}_{\alpha\in J}$ of g-fuzzy open sets in X is called a g- *open cover* of a fuzzy set μ in X if $\mu \subset \oplus_{\alpha \in J} \mu_\alpha$.

Definition 4.2 A fuzzy set κ of a g-fuzzy topological space X is called g-*fuzzy compact* if every g-open cover of κ has a finite g-open sub cover.

Definition 4.3 A g-fuzzy topological space X is called g-*fuzzy compact* if every g- open cover of X has a finite g-sub cover.

Definition 4.4 A g-fuzzy topological space (X,δ) is said to be locally fuzzy compact if each point $x \in X$ there exists a member $\mu \in \delta$ such that $x \in \mu$ and μ is g-fuzzy compact

Theorem 4.5 if κ is g-compact and σ is g-closed in a g-fuzzy topological space X and $\sigma \subset \kappa$, then σ is g-compact.

Proof :

Let $\{\mu_\alpha\}_{\alpha\in J}$ be a family of g-fuzzy open sets in X such that $\sigma \subset \oplus_{\alpha\in J} \mu_\alpha$.Then $\sigma^c \oplus (\oplus_{\alpha \in J} \mu_\alpha)$ covers X and hence there is a finite collection $\{\mu_{\alpha i}\}$ such that

$$\kappa \subset \sigma^c \oplus \mu_{\alpha 1} \oplus \ldots \oplus \mu_{\alpha n}.$$

Then, $\sigma \subset \mu_{\alpha 1} \oplus \ldots \oplus \mu_{\alpha n}$.

Hence σ is g-fuzzy compact.

Theorem 4.6 Let (X,δ) be a compact space and σ be a g- closed fuzzy subset of X. Then, σ is also a g-fuzzy compact space.

Proof:

Choosing a g-fuzzy open cover $\{\mu_\alpha\}_{\alpha\in J}$ for σ from δ and consider $\{\mu_\alpha\}_{\alpha\in J} \oplus \sigma^c$ a g- open cover for X. use the compactness of X to find a finite g-sub cover of σ .

Theorem 4.7. Every g-compact fuzzy subset of a Hausdorff g-fuzzy topological space is g-fuzzy closed.

Proof :

Let κ be a g-fuzzy compact subset of a Hausdorff g- fuzzy topological space X. Take any point $x \in \kappa^c$. We have to prove the existence of an g-fuzzy open set μ such that $x \in \mu \subset \kappa^c$. For any point $y \in \kappa$ use the Hausdorff property to find g-fuzzy open sets μ_y and ν_y with $x \in \mu_y$ and $y \in \nu_y$ and μ_y & ν_y = $\underline{0}$. Now $\{ \nu_y : y \in \kappa \}$ is a g- open cover for the g-fuzzy compact space κ will the existence of a finite g-sub cover , say ν_{y1} , ν_{y2} ,…, ν_{yn} for κ . Take $\mu = \nu_{y1}$& ν_{y2} &…& ν_{yn} .

Then μ is an g-fuzzy open subset containing x and $\mu \subset \kappa^c$.

Remark: If a g-fuzzy topological space (X,δ) is not Hausdorff then a fuzzy compact subset need not be closed.

Cor. If σ is g-closed and κ is g-compact in a Hausdorff g-fuzzy topological space (X,δ), then σ & κ is g-fuzzy compact

Proof:

As κ is a g-compact fuzzy subset of X, it is g-closed, by theorem, σ is g-closed implies σ & κ is g-closed fuzzy subset ofκ. By theorem, σ & κ is g- fuzzy compact.

Theorem 4.8 g**-** continuous image of a g- compact space is g-compact

Proof :

Let (X,δ) be a g-compact fuzzy topological space and (Y,η) be a g-fuzzy topological space. Given f : $(X,\delta) \rightarrow (Y,\eta)$ is g-fuzzy continuous function .

The g-open cover $\{\mu_\alpha\}_{\alpha \in J} \subset \eta$ for f (X) will produce an g-open cover

$\{f^{-1}(\mu_\alpha)\}_{\alpha\in J} \subset \delta$ for X. Use the g-fuzzy compactness of X, to find a finite g-sub cover of η for f (X).

Theorem 4.9 If f : (X, δ) $\rightarrow$ (Y, η) is g-fuzzy continuous function and κ is a g- fuzzy compact set in (X, δ) then f (κ) is g-fuzzy compact in (Y, η).

Proof:

Let $\{\mu_\alpha\}_{\alpha \in J}$ be a g- open cover of f(κ).

Then, $\{f^{-1}(\mu_\alpha)\}_{\alpha \in J}$ is a g- open cover of κ.

As κ is g-fuzzy compact, $\kappa \subset f^{-1}(\mu_{\alpha 1}) \oplus f^{-1}(\mu_{\alpha 2}) \oplus \ldots \oplus f^{-1}(\mu_{\alpha n})$

for some $\alpha_1, \alpha_2, \ldots, \alpha_n$ and therefore $f(\kappa) \leq \mu_{\alpha 1} \oplus \mu_{\alpha 2} \oplus \ldots \oplus \mu_{\alpha n}$. Hence f($\kappa$) is g- fuzzy compact in (Y,η).

Remark: If A , B $\in$ P(X) , then $A \oplus B = A \cup B$, A& B = $A \cap B$ and AϴB = A \ B. Thus the ordinary topology and ordinary topological spaces become special cases of g- fuzzy topology and g- fuzzy topological spaces.

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